New Classes of Distributed Time Complexity

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Based on

- **New Classes of Distributed Time Complexity** - Alkida Balliu, Juho Hirvonen, Janne H. Korhonen, Tuomo Lempiäinen, Dennis Olivetti, and Jukka Suomela [STOC'18]
- **Almost Global Problems in the LOCAL Model** - Alkida Balliu, Sebastian Brandt, Dennis Olivetti, and Jukka Suomela [Submitted]

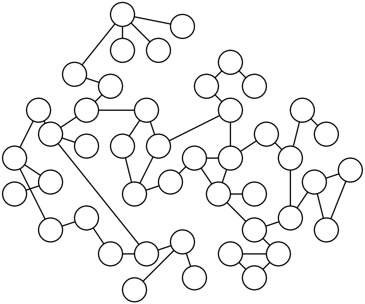
Slides based on "New Classes of Distributed Time Complexity", Janne H. Korhonen

Outline

- LOCAL Model
- Locally Checkable Labellings
- Results
- Proof idea

LOCAL Model

- Distributed
- Unlimited bandwidth
- Unlimited computational power
- Nodes have IDs
- In this talk:
  - deterministic algorithms



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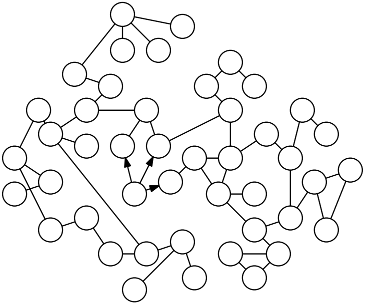

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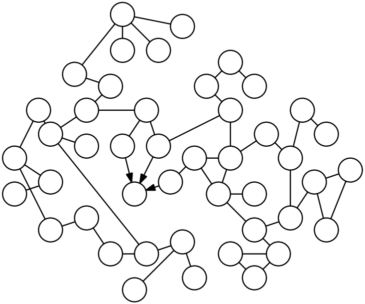

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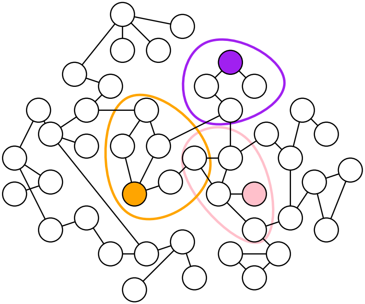
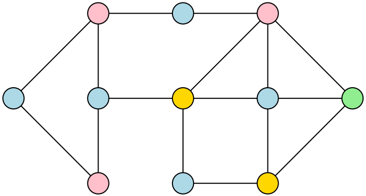


Locally Checkable Labellings

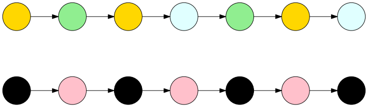
LCL Problems:

- Introduced by Naor and Stockmeyer in 1995
- Constant-size input labels
- Constant-size output labels
- The maximum degree is constant
- Validity of the output is locally checkable



LCL on Cycles

- There are only three possible time complexities:
  - $\Theta(1)$: trivial problems
  - $\Theta(log^* n)$: local problems (symmetry breaking)
  - $\Theta(n)$: global problems



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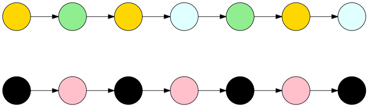
- Automatic speedups:
  - Any $o(\log^* n)$-rounds algorithm can be converted to a $O(1)$-rounds algorithm [Naor and Stockmeyer, 1995]
  - Any $o(n)$-rounds algorithm can be converted to a $O(\log^* n)$-rounds algorithm [Chang, Kopelowitz and Pettie, 2016]



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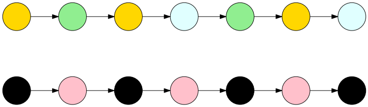
- On cycles with no input, given an LCL description, we can *decide* its time complexity. [Naor and Stockmeyer, 1995] [Brandt et al, 2017]



LCL on Cycles

LCL on Trees

[Chang and Pettie, 2017]:

- Any $n^{o(1)}$-rounds algorithm can be converted to a $O(\log n)$-rounds algorithm
- There are problems of complexity $\Theta(n^{1/k})$

LCL on Trees

LCL on Trees (Our Results)

LCL on General Graphs

- There are problems with complexity $\Theta(\log n)$ [Brandt et al, 2016] [Chang, Kopelowitz and Pettie, 2016] [Ghaffari and Su, 2017]

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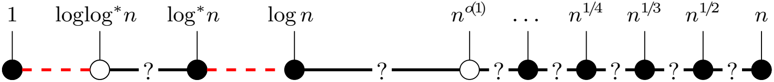
- Many problems require $\Omega(\log n)$ and $O(\text{poly} \log n)$
- Different scenario with randomized algorithms

LCL on General Graphs



LCL on General Graphs?

LCL on General Graphs (Our Results)

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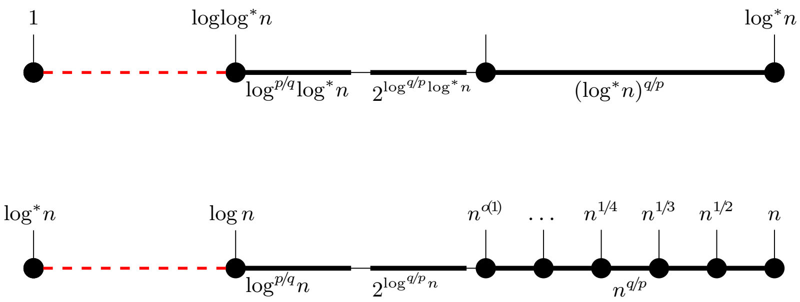


General Idea

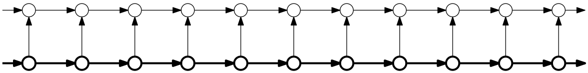
- We start from an LCL problem Π on cycles:
  - Π_l has complexity $T(n) = \Theta(\log^* n)$
    - 3 colouring
  - Π_g has complexity $T(n) = \Theta(n)$
    - a variant of 2 colouring
- We build a speed-up construction:
  - in ℓ rounds a node "sees" at distance $f(\ell) = \ell g(\ell)$
  - we obtain an easier version of Π
  - new complexity:
    - $\Theta(f^{-1}(T(n)))$

Example

- We start from a cycle

Example

- We add shortcuts on top of the cycle, $g(\ell) = 2^\ell$



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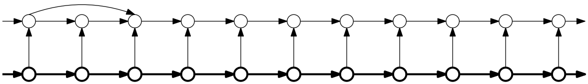

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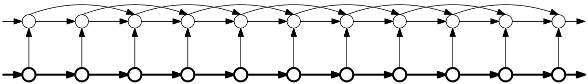

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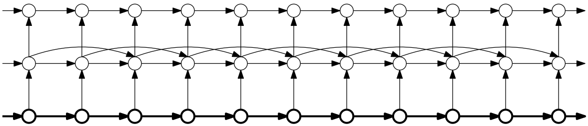

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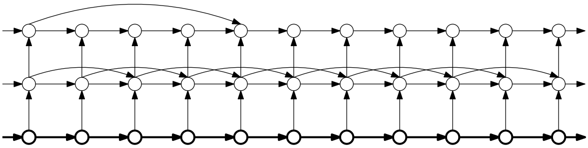

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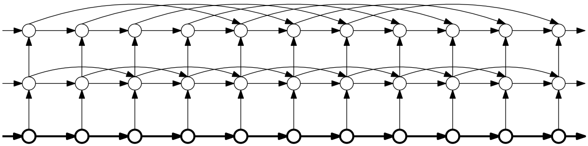
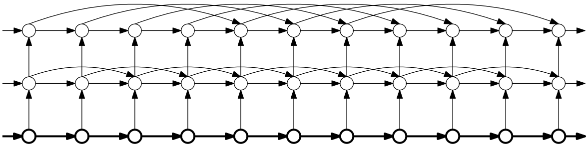


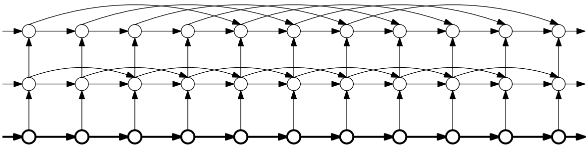
Example

- Π can be solved in $\Theta(f^{-1}(T(n)))$ rounds using the shortcuts



Example

- Problem: this is not a valid LCL



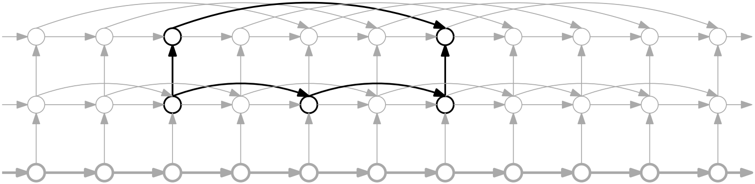
Valid LCL

- An LCL problem must be defined on any graph, not just on some "relevant" instances
- What if the shortcuts are missing?
- What if a cycle is not present at all?

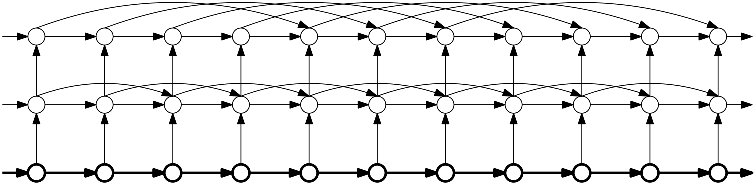
Fixing the details

- Input:
  - a graph
  - a proof that the graph is a relevant instance
    - it must be locally checkable
- Output:
  - Solve Π, or
  - Prove that there is an error in the input proof, or in the graph structure
    - it must be locally checkable

Local checkability of the input



Correct instance



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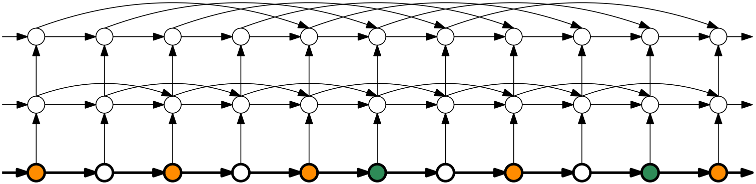

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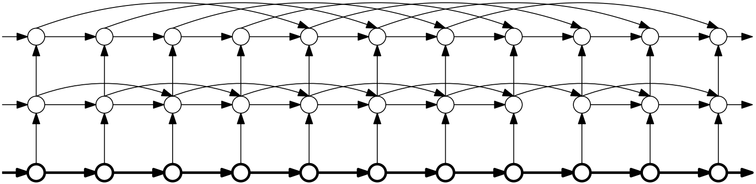

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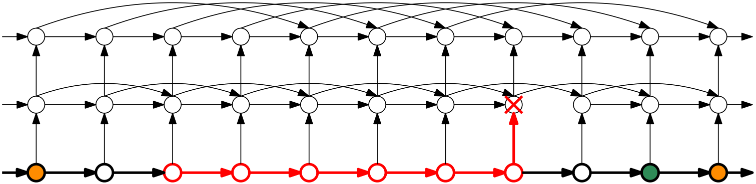
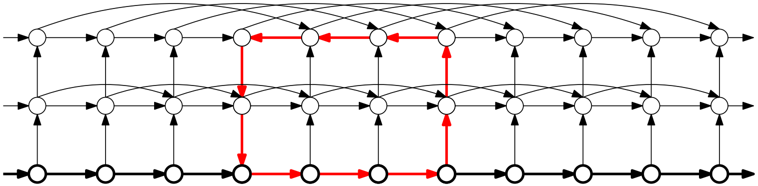
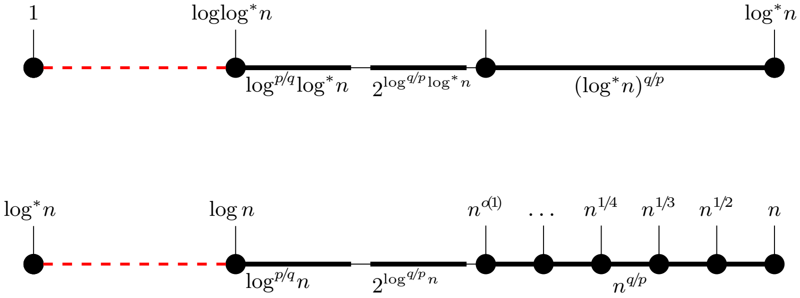


Hardness balance

- On incorrect instances, it should be easy to prove that there is an error
- On correct instances, it should be impossible, or hard, to prove that there is an error



Using different $g(\ell)$



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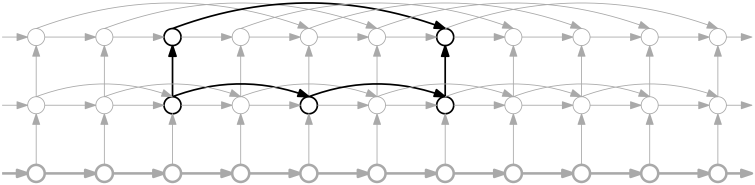


Which shortcut constructions can be locally checked?

Link Machine Programs

- Constant number of registers
- Reset
  - $r_1 \leftarrow 1$
- Addition
  - $r_1 \leftarrow r_2 + r_3$
- If statements with equality comparison
  - if $r_1 = r_2$
  - if $r_1 \neq r_2$
- $g(\ell)$ = value of the maximum register after ℓ executions of the program

Link Machine Programs: Examples

- $g(\ell) = 2^\ell$
  - $r_1 \leftarrow r_1 + r_1$

Link Machine Programs: Examples

- $g(\ell) = 2^\ell$
  - $r_1 \leftarrow r_1 + r_1$
- $g(\ell) = \Theta(\ell^2)$
  - $r_1 \leftarrow r_1 + 1$
  - $r_2 \leftarrow r_2 + r_1$

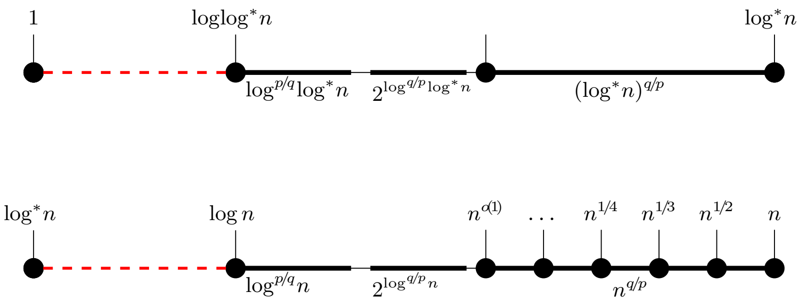
Link Machine Programs: Building Blocks

Program P	Input	Output	Growth
COUNT	–	$y = \ell$	ℓ
ROOT'_k	–	$y = \Theta(\ell^{1/k})$	$\Theta(\ell^{1/k})$
ROOT_k	x	$y = \Theta(x^{1/k})$	$\Theta(x)$
POW_k	x	$y = \Theta(x^k)$	$\Theta(x^k)$
EXP	x	$y = 2^{\Theta(x)}$	$2^{\Theta(x)}$
LOG	x	$y = \Theta(\log x)$	$\Theta(x)$

Link Machine Programs: $g(\ell)$

Program P		Growth
$\mathrm{POW}_p \circ \mathrm{ROOT}'_q$		$\Theta(\ell^{p/q})$
$\mathrm{EXP} \circ \mathrm{POW}_q \circ \mathrm{ROOT}'_p$	$(p \geq q)$	$2^{\Theta(\ell^{q/p})}$
$\mathrm{EXP} \circ \mathrm{POW}_q \circ \mathrm{ROOT}_p \circ \mathrm{LOG} \circ \mathrm{COUNT}$	$(p \geq q)$	$2^{\Theta(\log^{q/p} \ell)}$

Results



1
$\log\log^* n$
$\log^* n$
$\log^{p/q}\log^* n$
$2^{\log^{q/p}\log^* n}$
$(\log^* n)^{q/p}$
$\log^* n$
$\log n$
$n^{o(1)}$
$\ldots$
$n^{1/4}$
$n^{1/3}$
$n^{1/2}$
n
$\log^{p/q} n$
$2^{\log^{q/p} n}$
$n^{q/p}$

Conclusions and Open Problems

- Are there other gaps on trees?
- What happens between $\Omega(\log \log^* n)$ and $O(\log^* n)$ on trees?
- What about polynomial complexities with sub-diameter time/sub-linear volume?
- What are meaningful subclasses of LCL problems where there are gaps again?

Thank you!