Distributed Edge Coloring in Time Polylogarithmic in Δ

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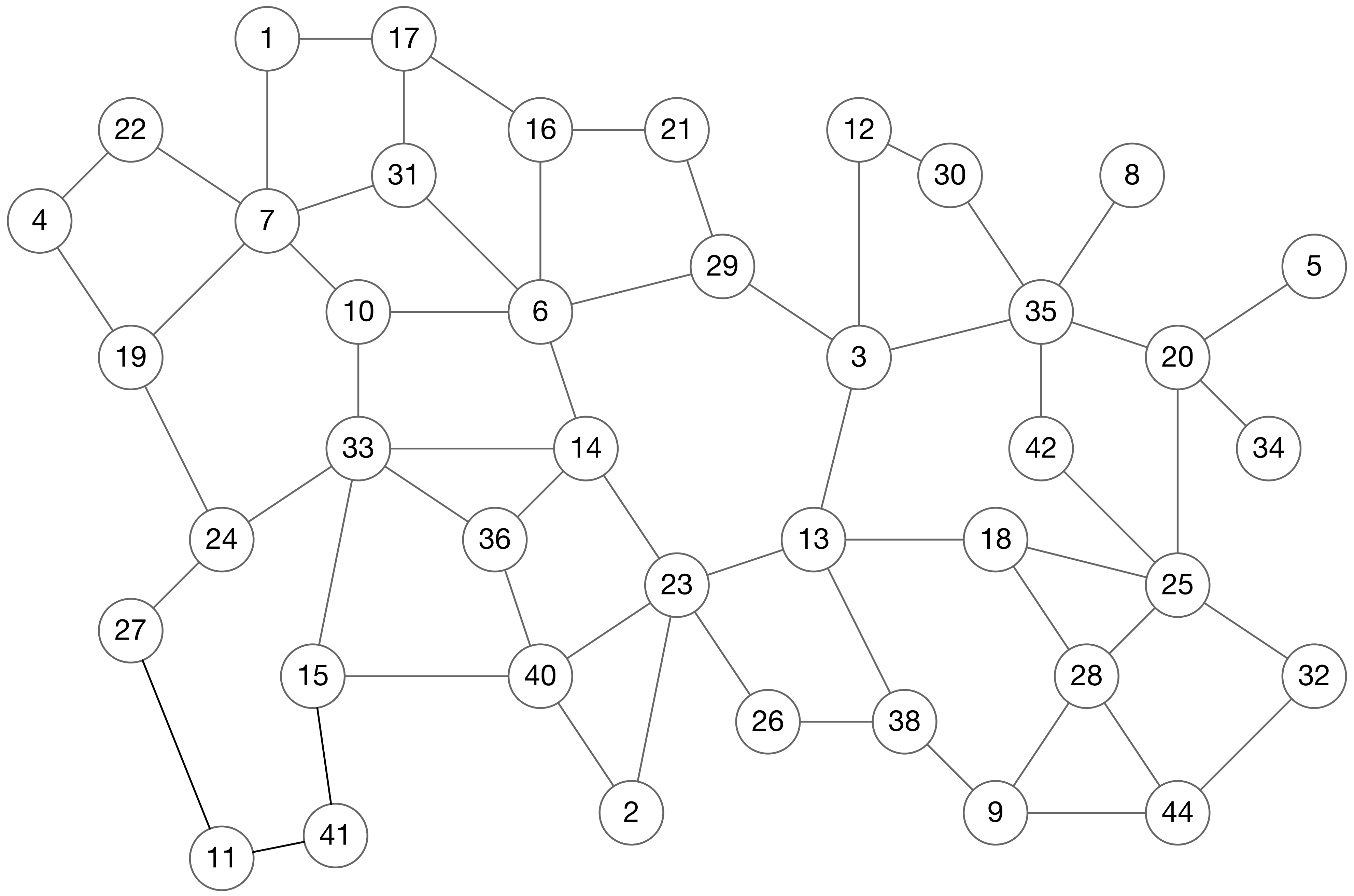
LOCAL model

- Undirected simple graph $G = (V, E)$ of n nodes and maximum degree Δ
- Each node has a unique ID
- Synchronous message passing model
- Unbounded computation
- Unbounded bandwidth



CONGEST model

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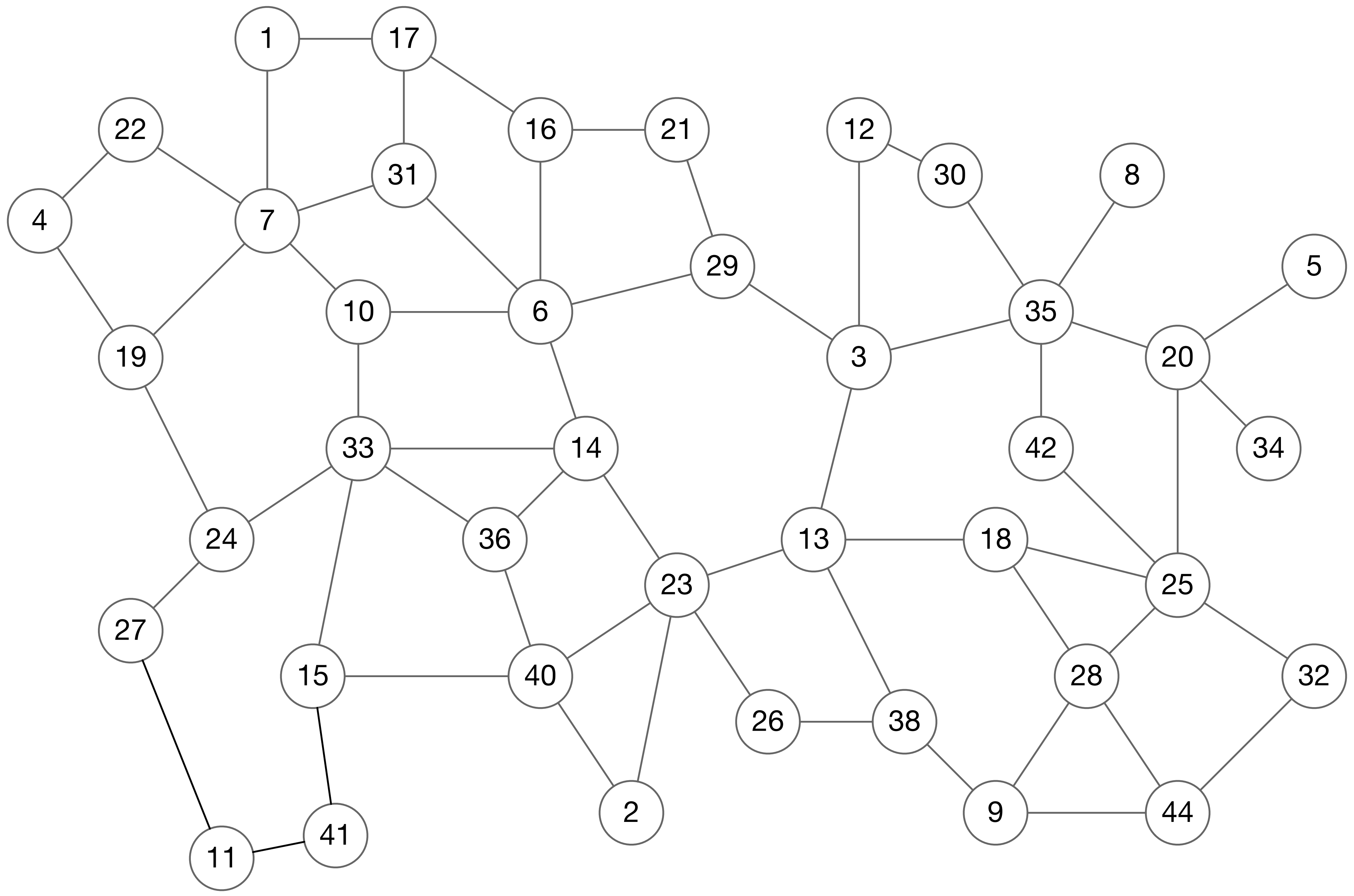
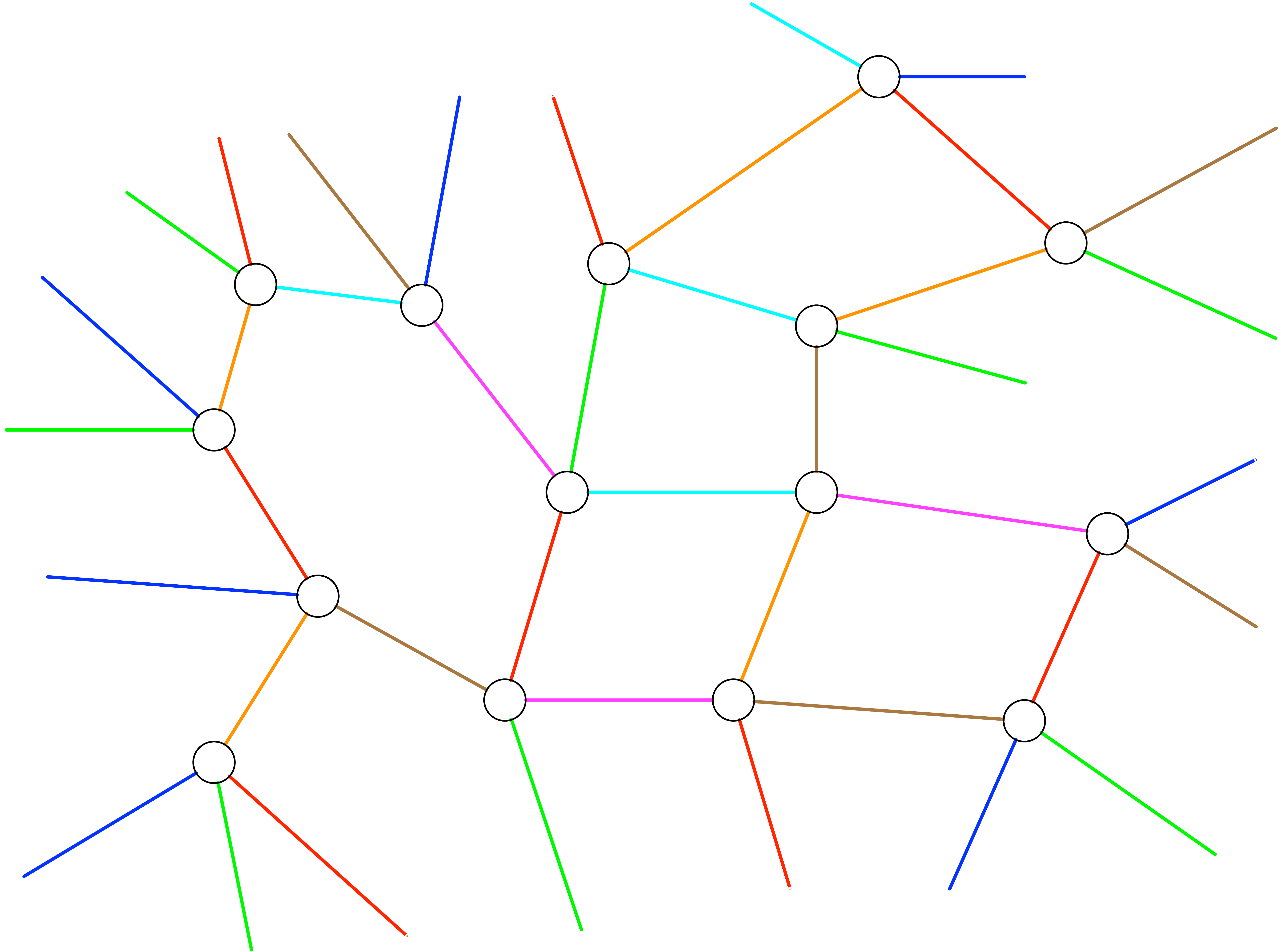
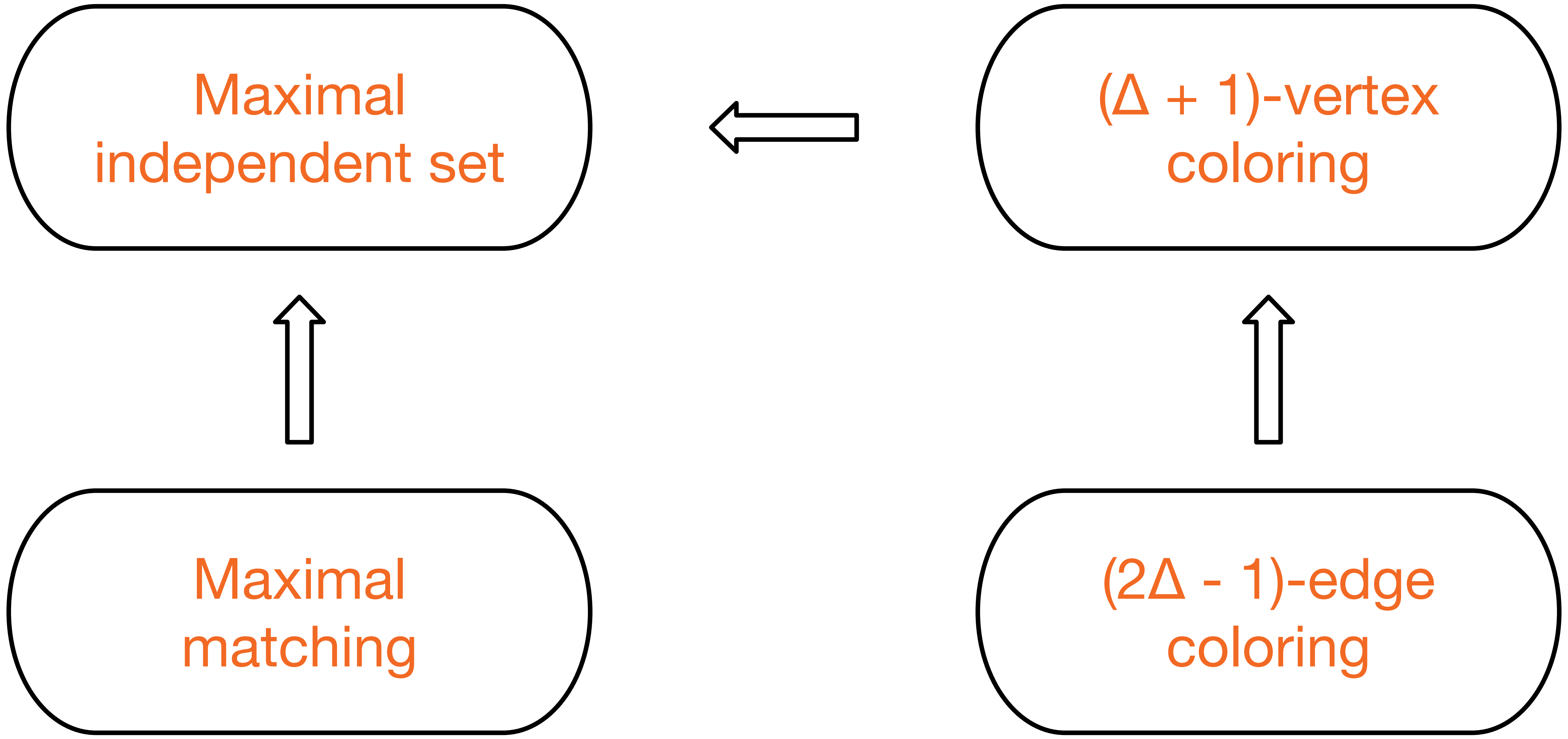
- Each node has a unique ID
- Synchronous message passing model
- Unbounded computation
- O(log n)-bit messages



(2Δ - 1)-Edge Coloring



Four classical problems



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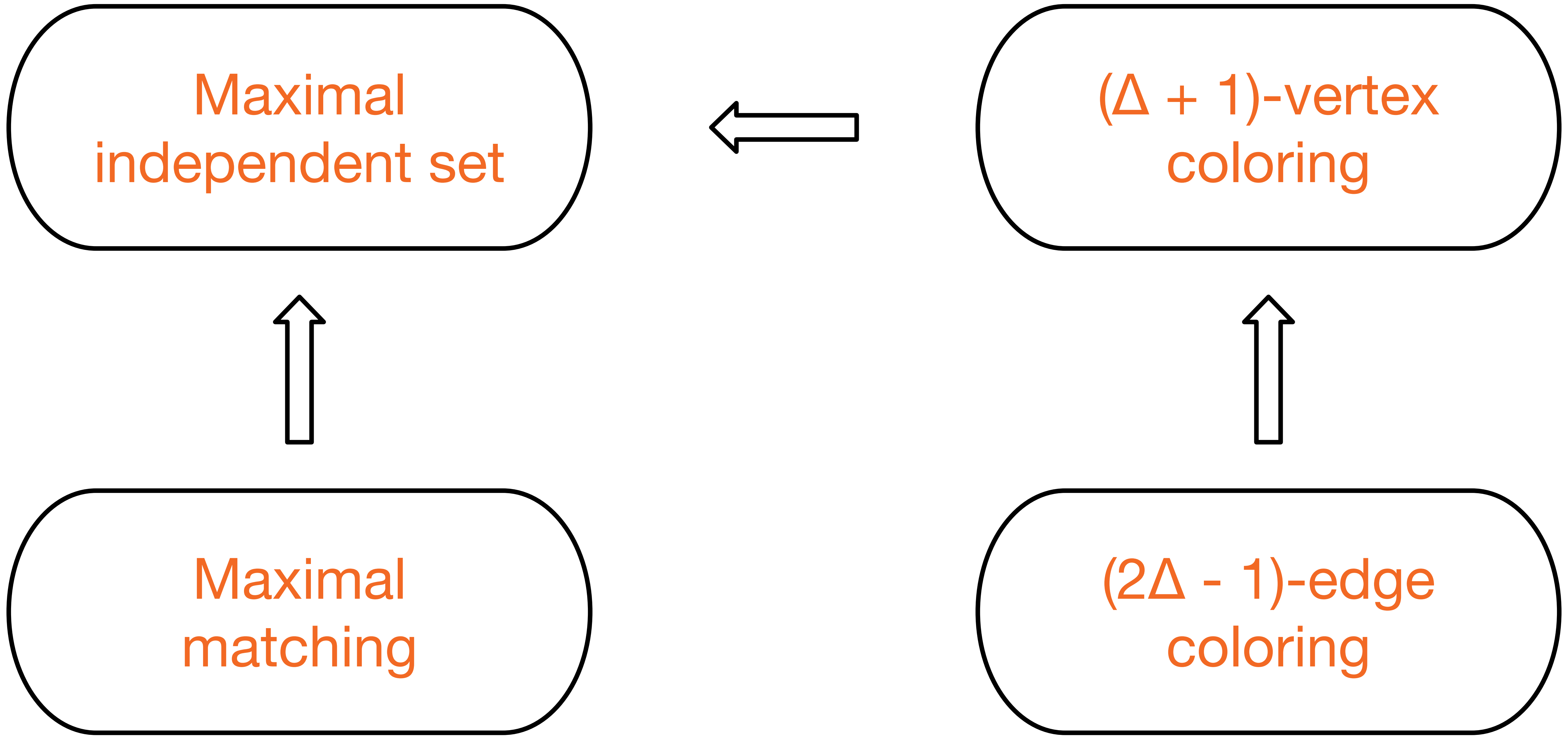

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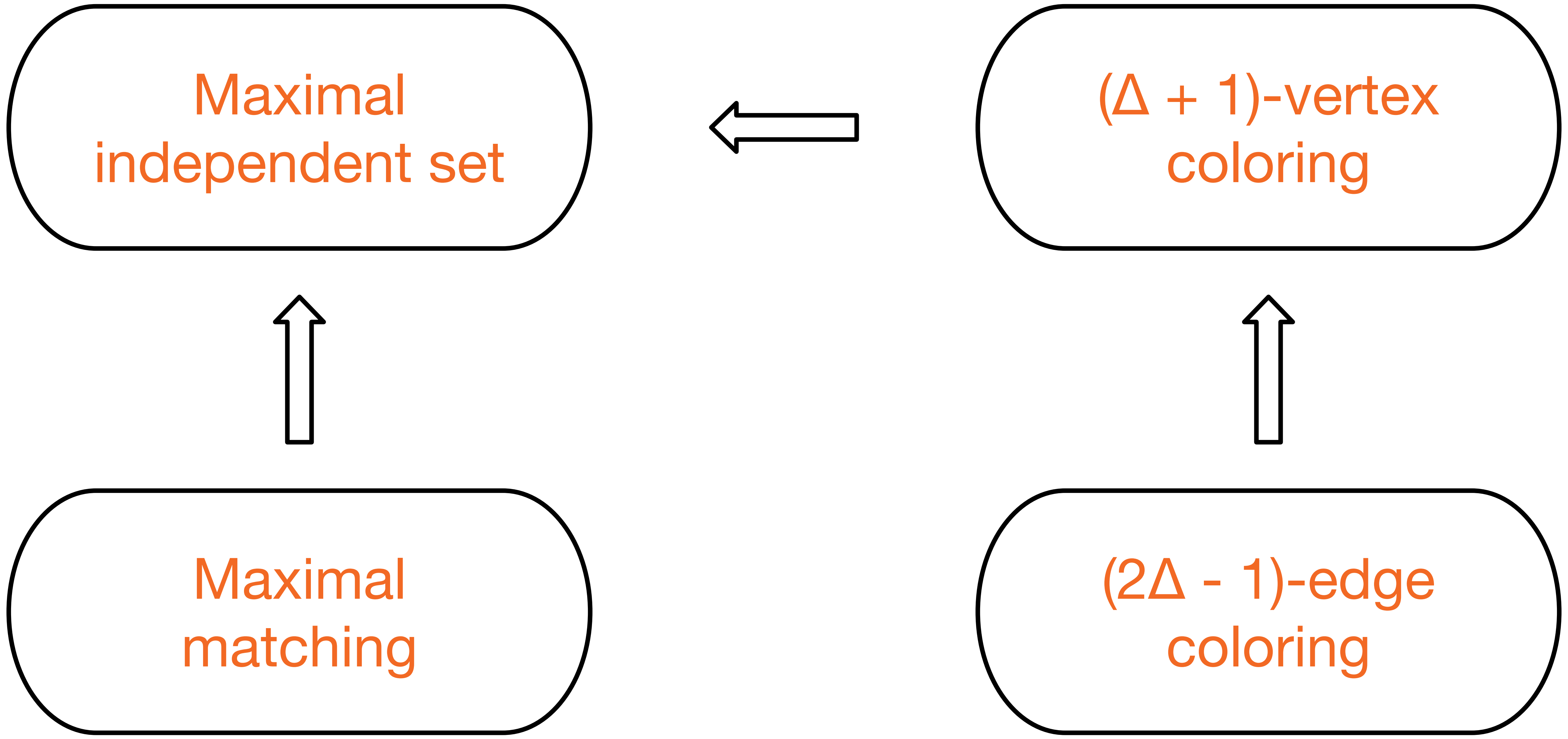

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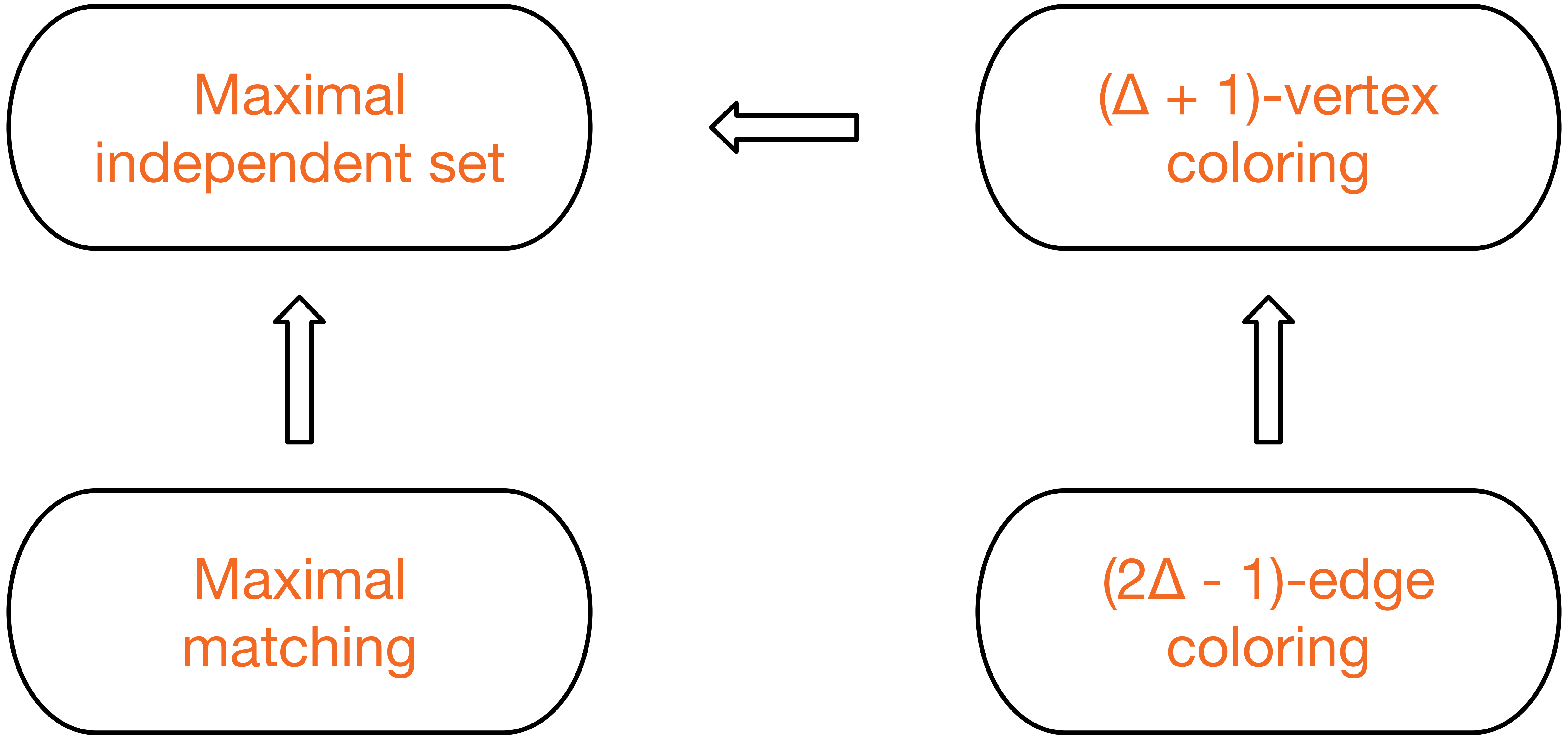

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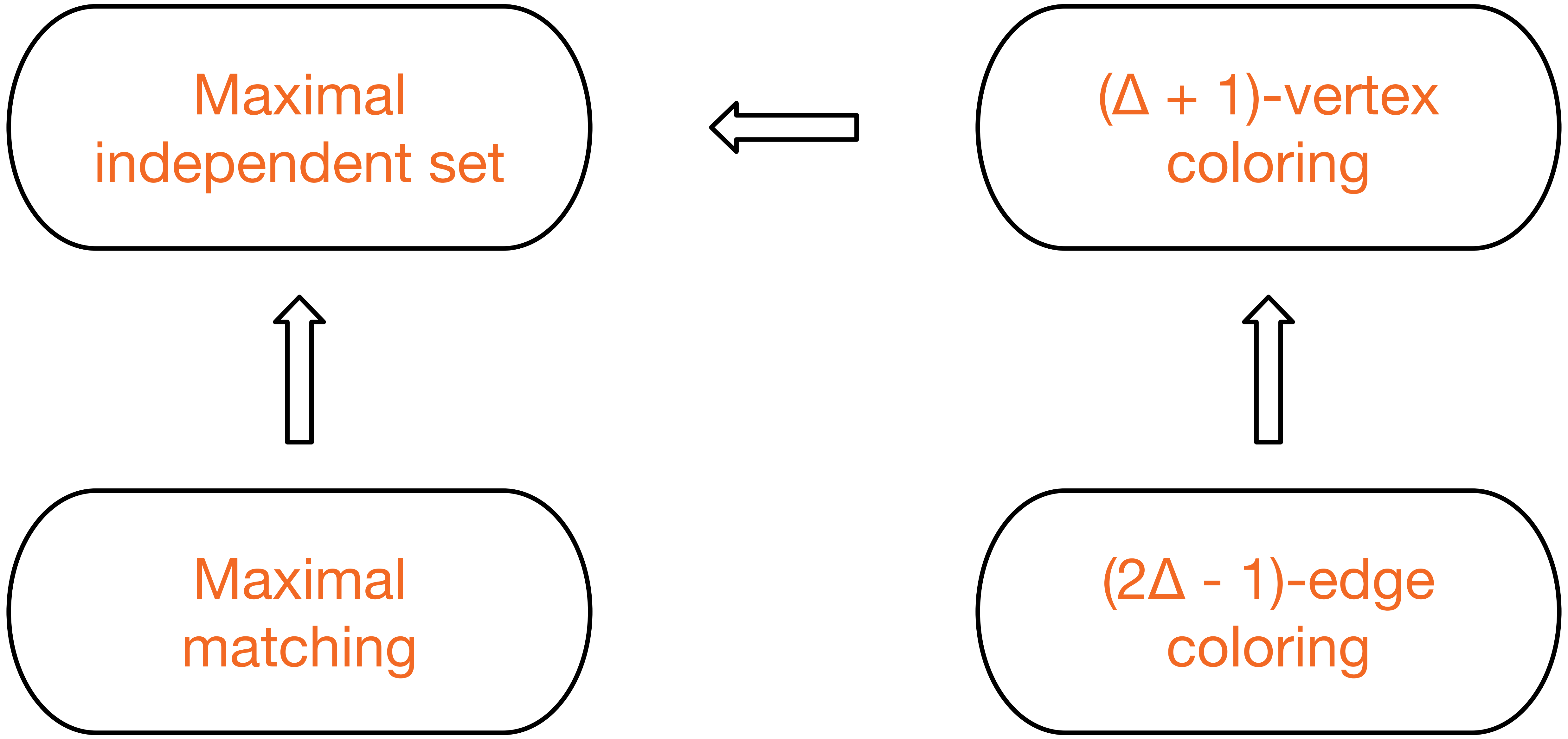

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Big question: $f(\Delta) + O(\log^* n)$

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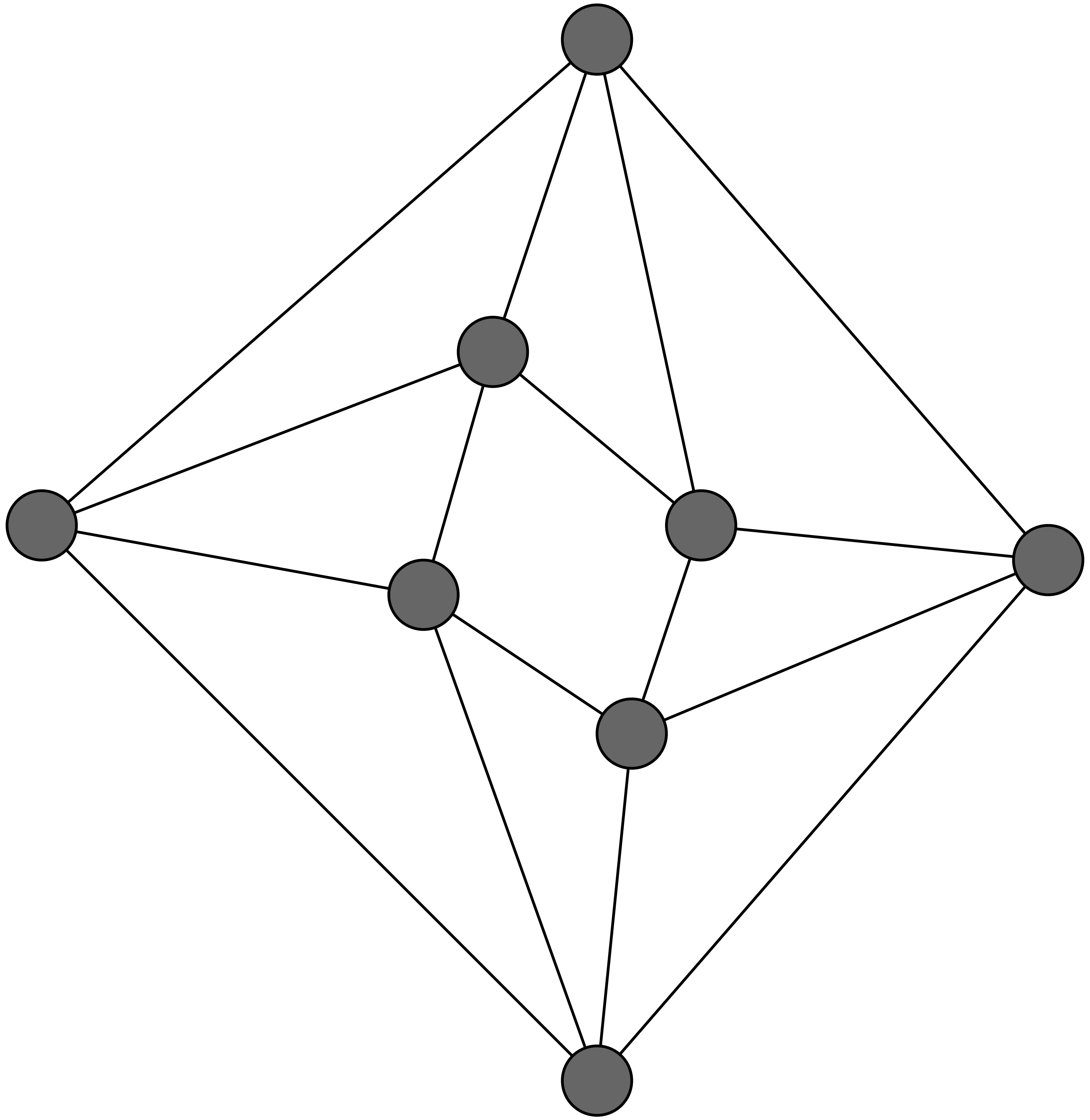
Our results

$(2\Delta - 1)$-Edge Coloring	$O(\text{poly} \log \Delta + \log^* n)$	LOCAL model
$O(\Delta)$-Edge Coloring	$O(\text{poly} \log \Delta + \log^* n)$	CONGEST model

Our results

$(\text{degree} + 1)$-List Edge Coloring	$O(\log^7 C \cdot \log^5 \Delta + \log^* n)$	LOCAL model
$(2\Delta - 1)$-Edge Coloring	$O(\log^{12} \Delta + \log^* n)$	LOCAL model
$(8 + \varepsilon)\Delta$-Edge Coloring	$O\left(\frac{\log^{12} \Delta}{\varepsilon^6} + \log^* n\right)$	CONGEST model

A possible approach



Try to **recursively color** the edges with **2 colors**, such that each node has roughly the **same amount of incident edges for each color**

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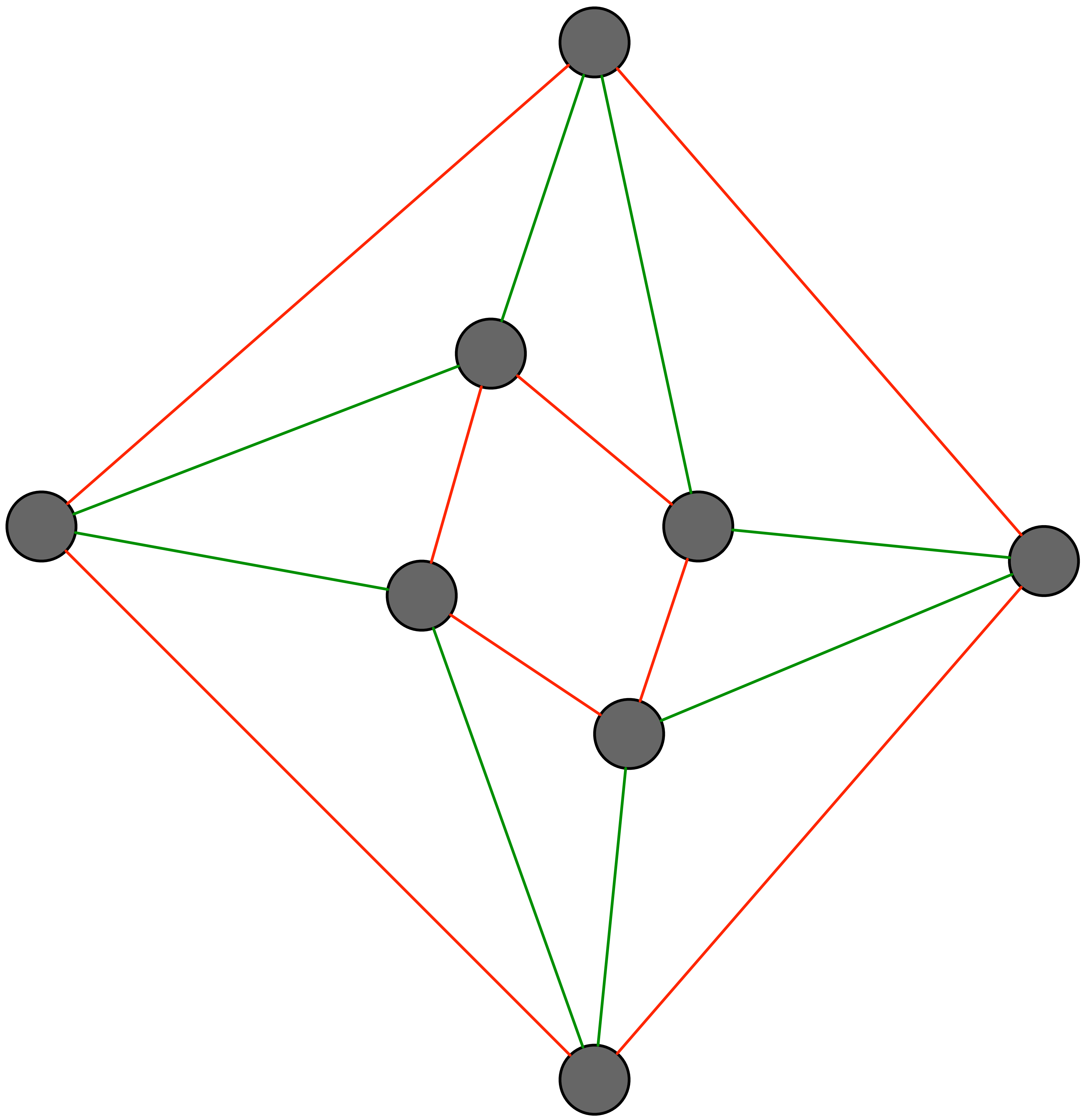

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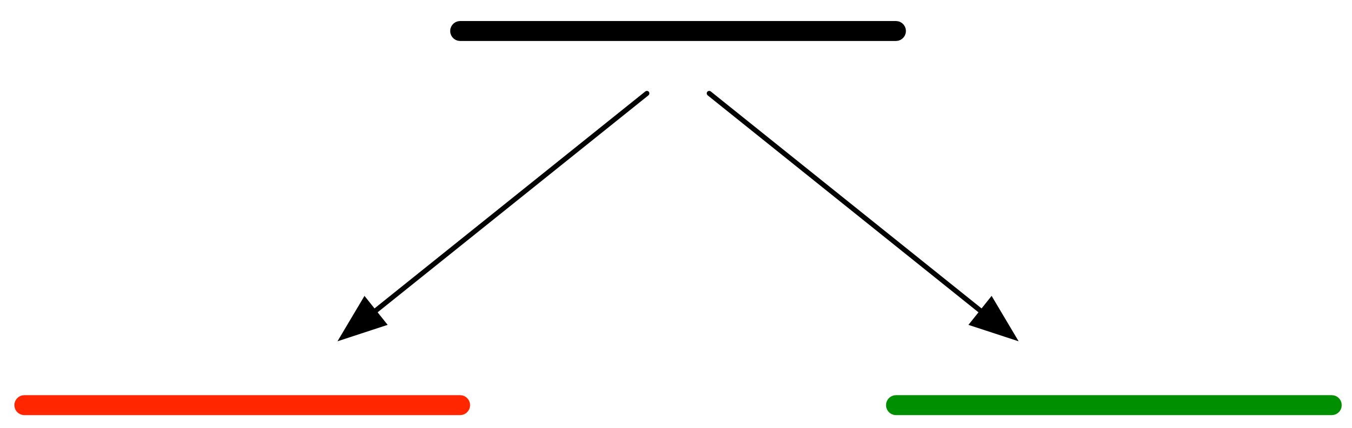

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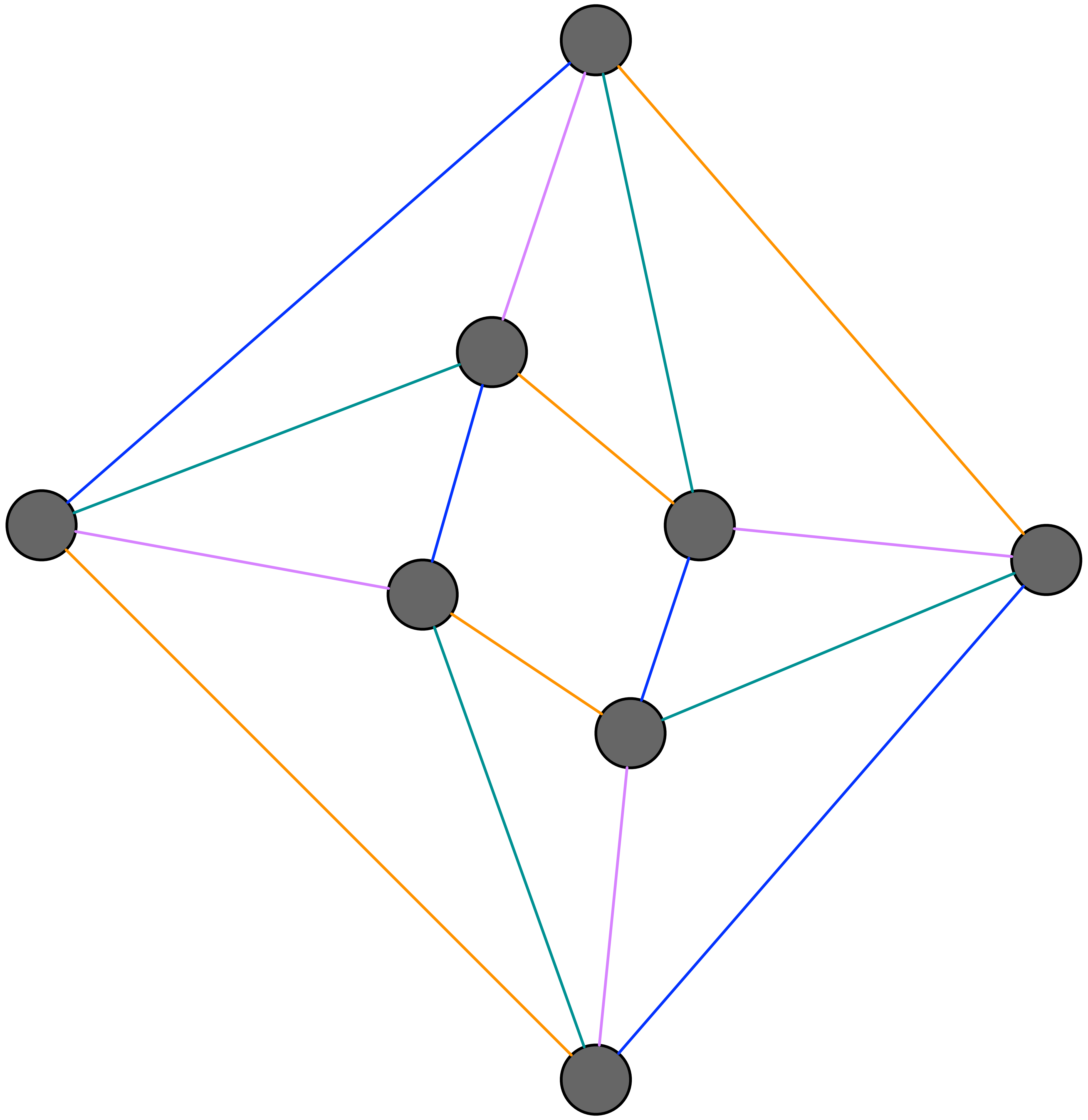

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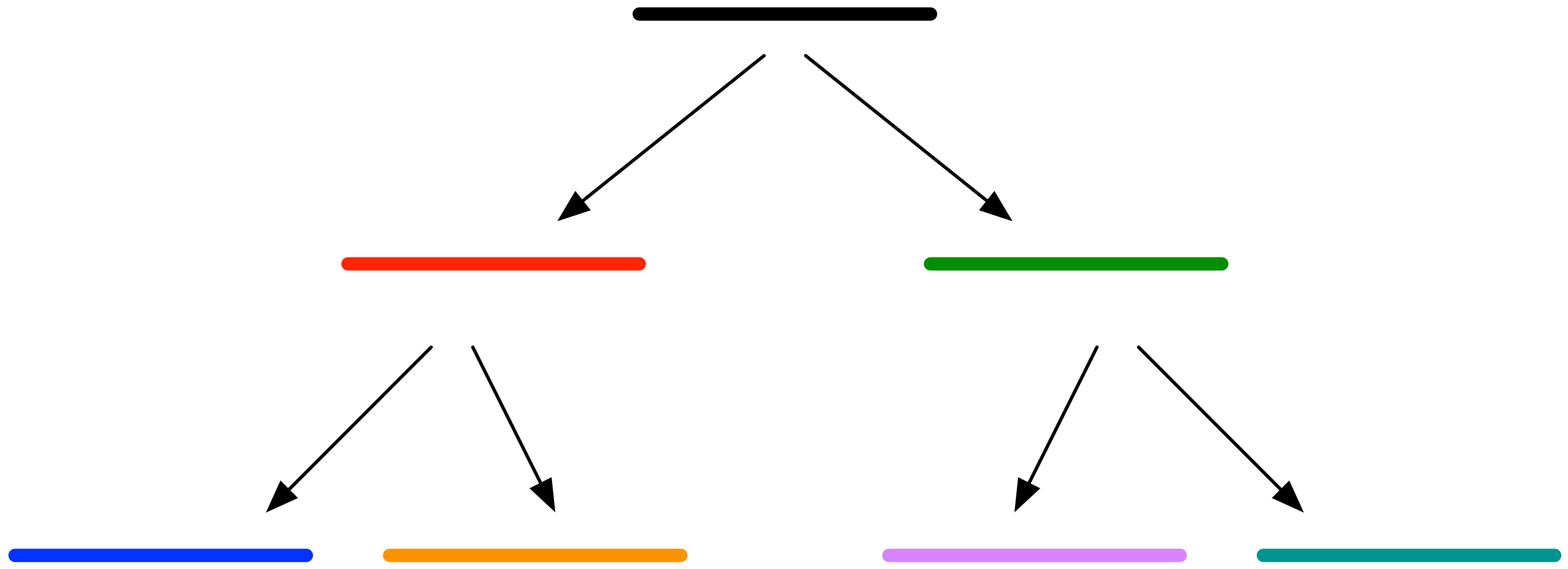

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This requires too much!

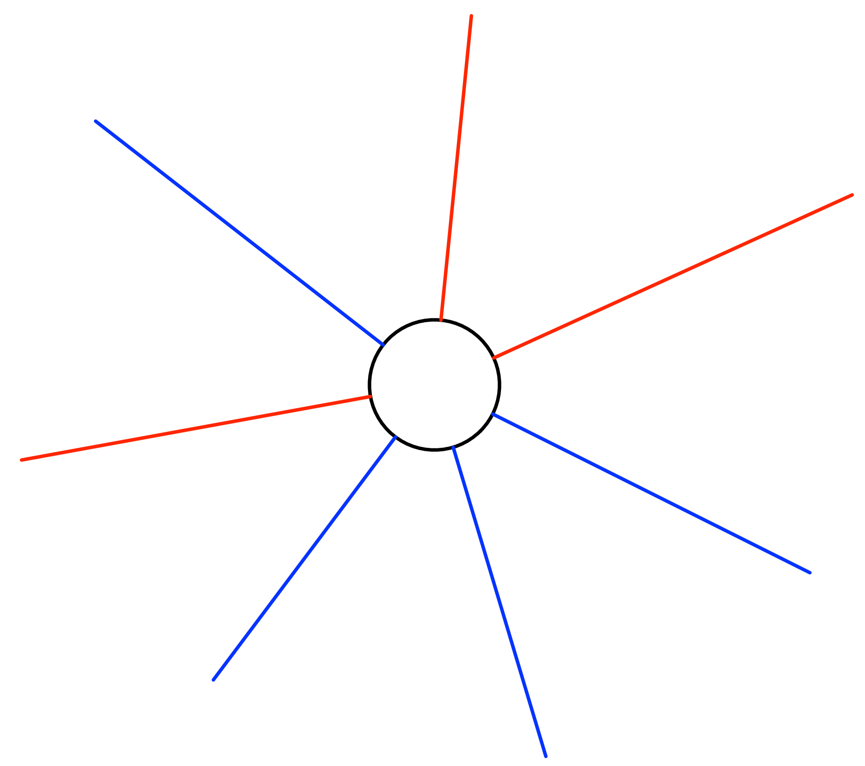
Can be done in just $O(\log^* n)$

A possible approach: the issue

- **d**-**node**-defective **2**-edge-coloring: **color the edges** with **2** colors such that **each node** has at most **d** incident **edges of the same color**.

  This is **hard,** even for $d \leq \Delta - 1.$

  It requires $\Omega(\log n)$ rounds!



A different subroutine

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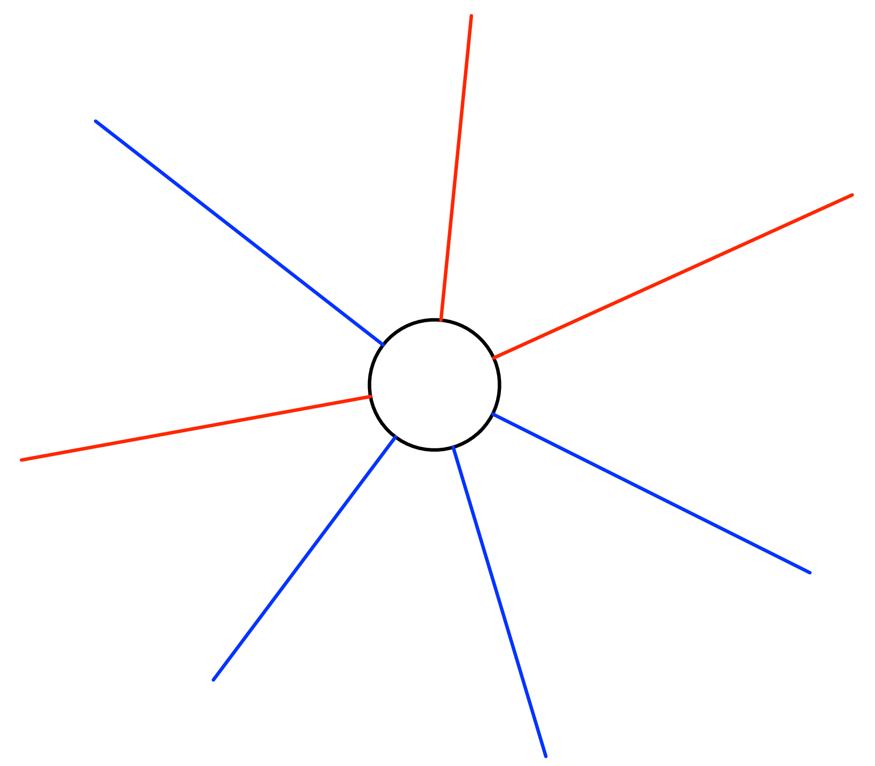

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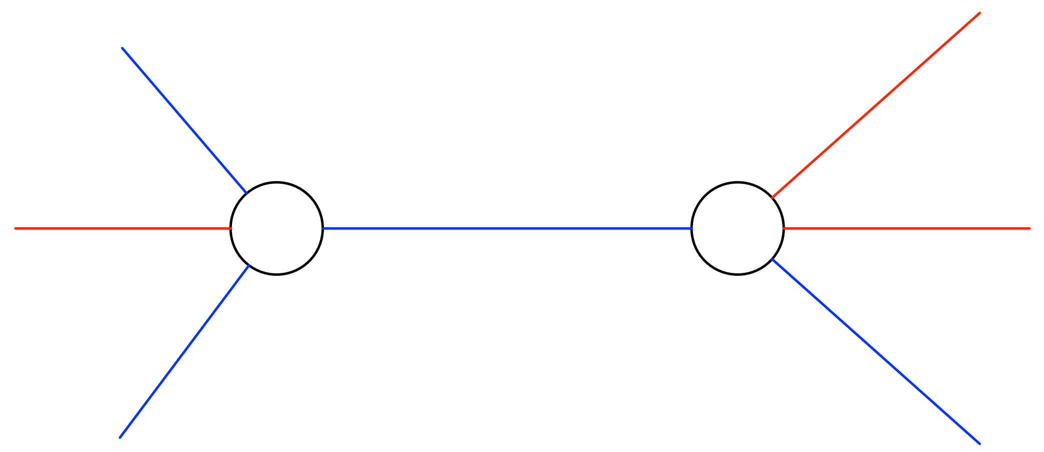
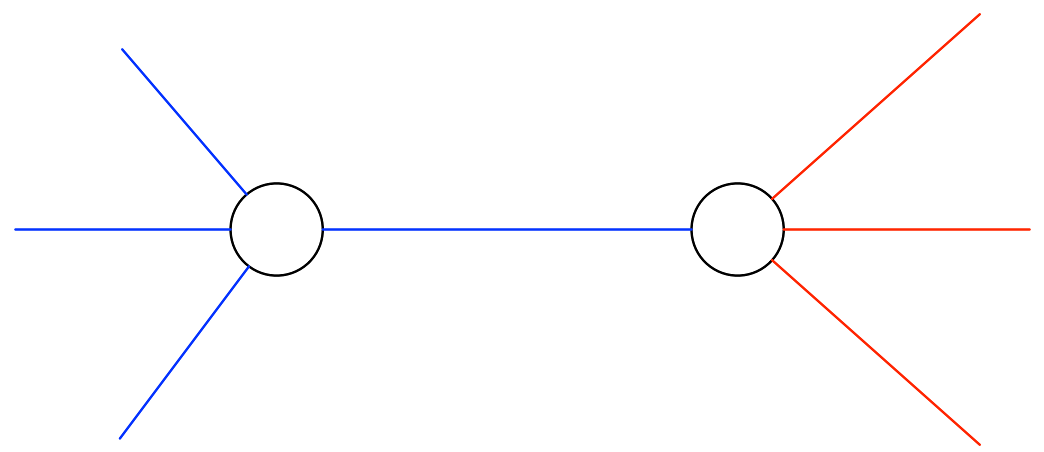


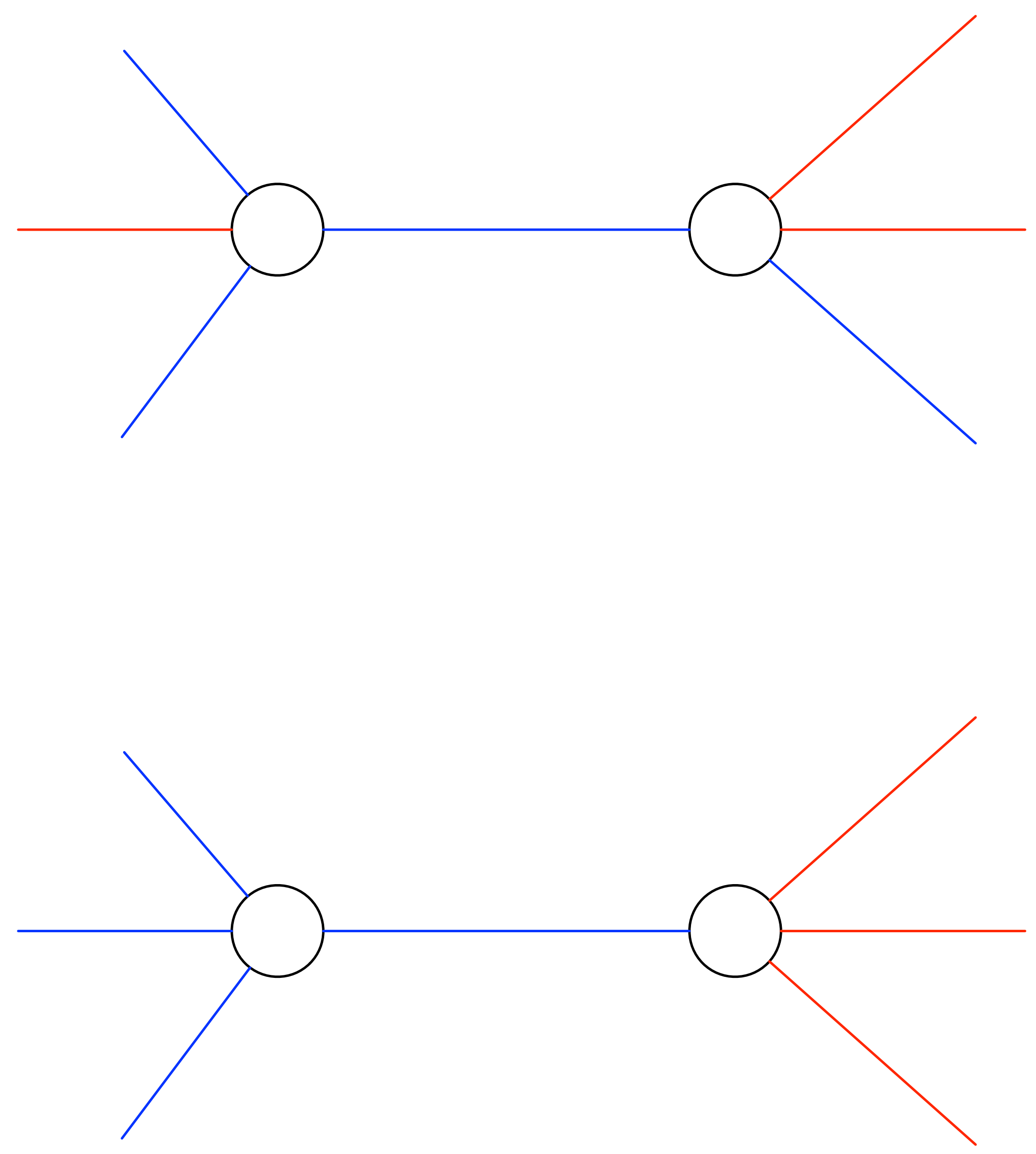
This is the problem that we tried to solve, for $d \leq (1 + \varepsilon)\Delta$



Main Ingredient

d-edge-defective **2**-edge-coloring:

> **color the edges** with **2** colors such that **each edge** has at most **d** incident **edges of its color**.

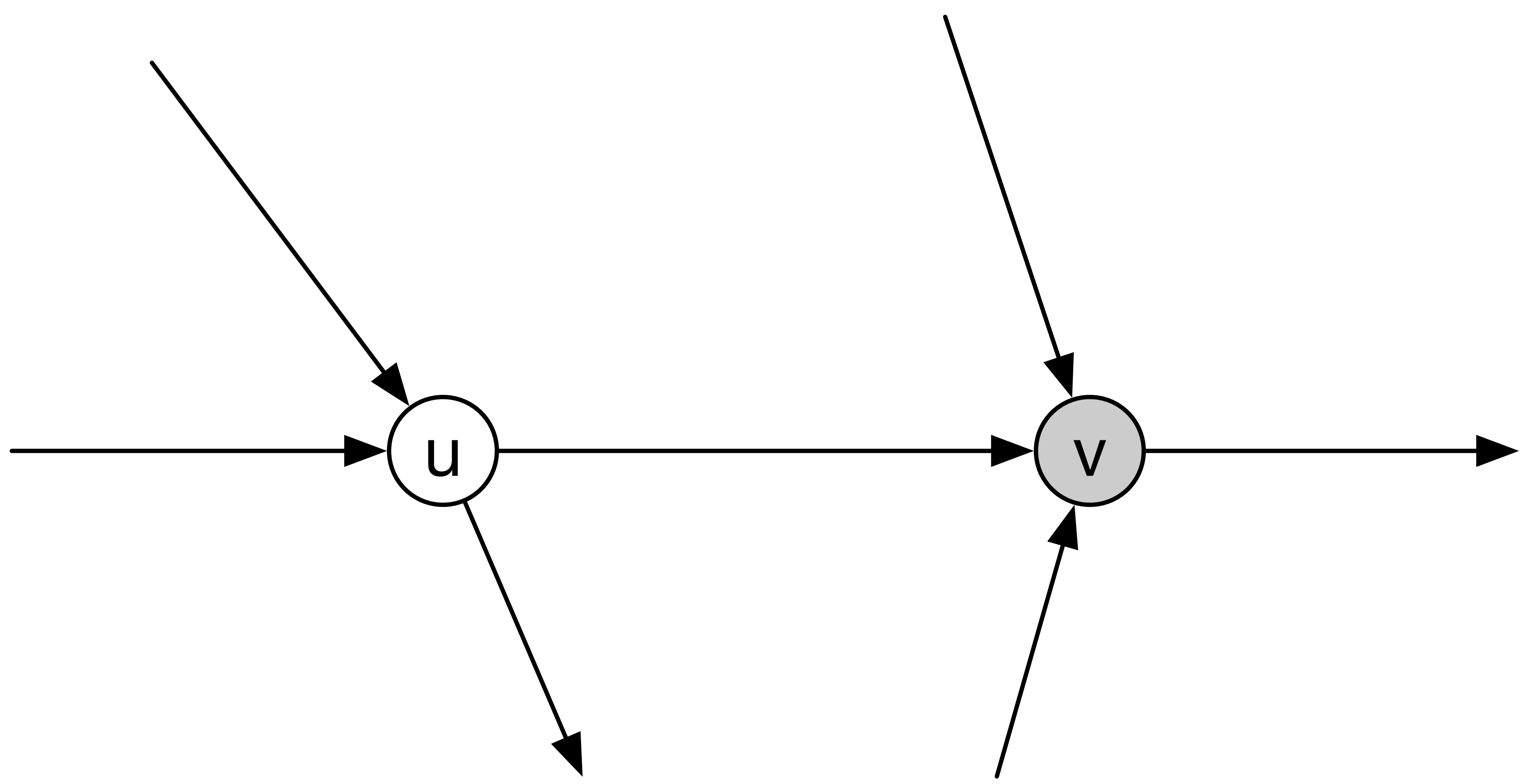


For $d \leq (1+\varepsilon)\Delta$, the problem can be solved in $O(\text{poly}(1/\epsilon, \log \Delta))$ time!

(for list coloring, we need a bit more)

Stable Orientation

- **Orient** the edges of a graph such that, for each edge (u, v) oriented from u to v, it holds that $\deg_{\text{in}}(v) \leq \deg_{\text{in}}(u) + 1$



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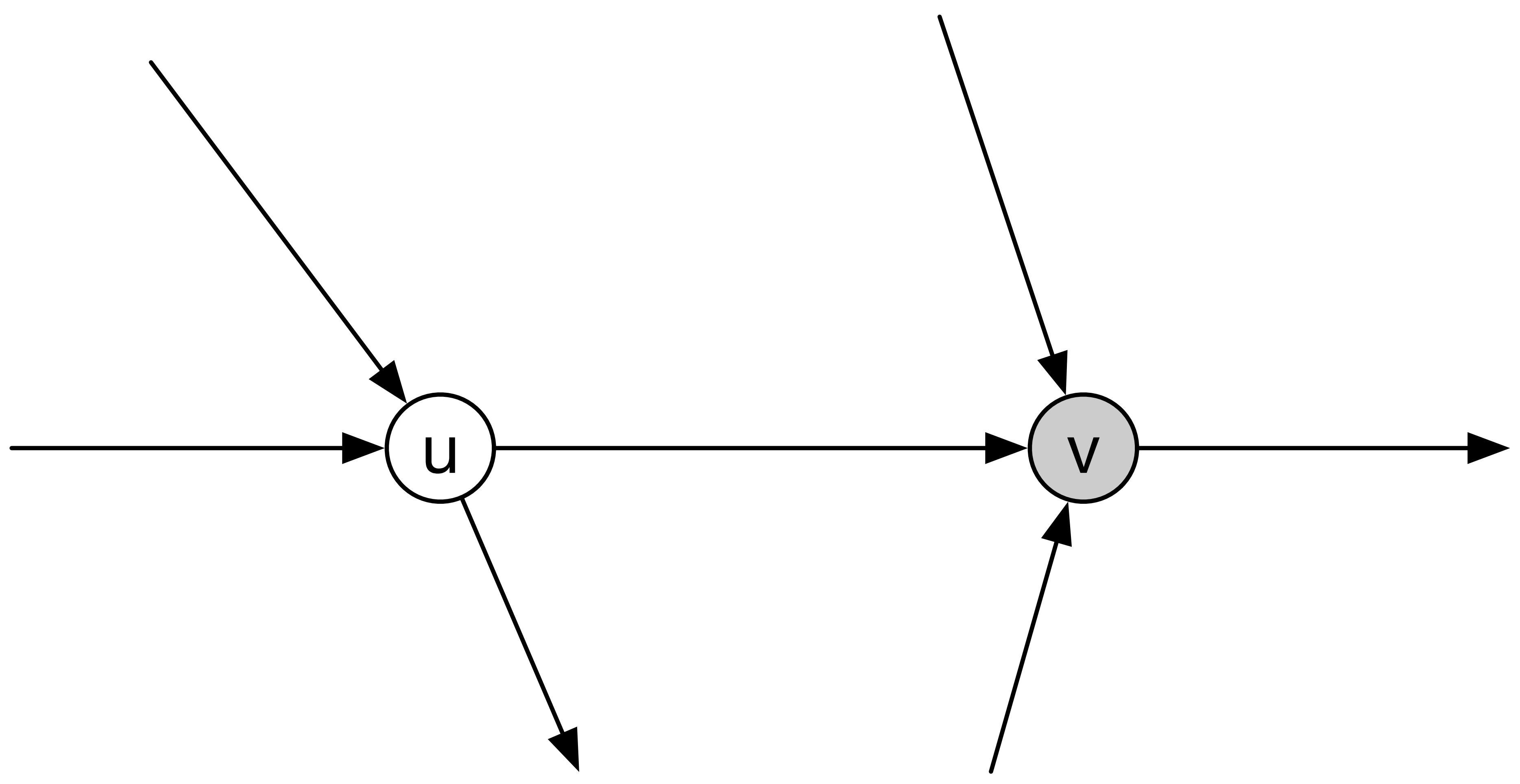

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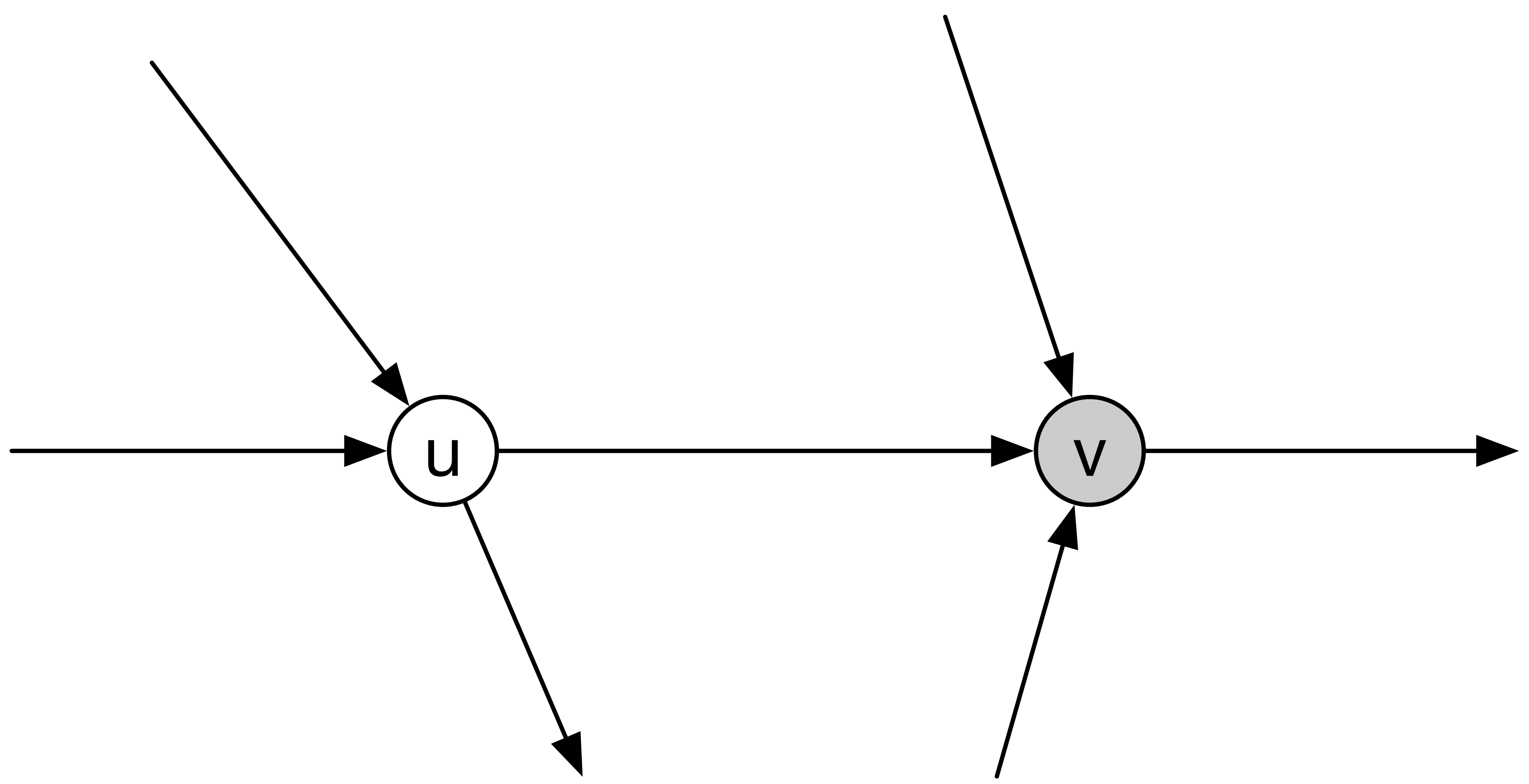

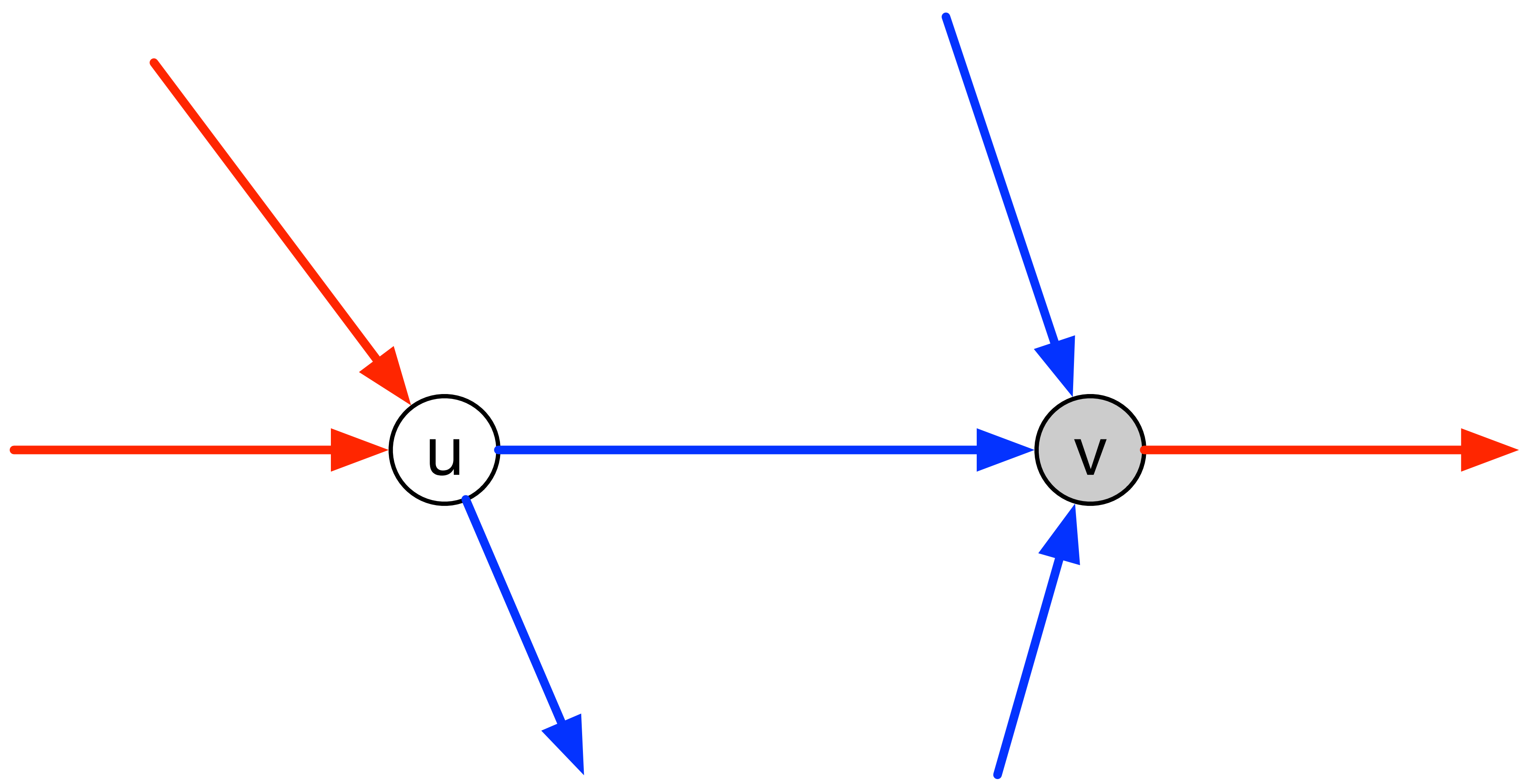

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Efficient Load-Balancing through Distributed Token Dropping

[Brandt, Keller, Rybicki, Suomela, Uitto 2021]

This problem can be solved in $O(\Delta^4)$ rounds!

Issues

- **Stable orientation** solves **"balanced" edge 2-coloring**, but:
  - The running time is $O(\Delta^4)$, we want $O(\log^c \Delta)$
  - We can turn a stable orientation into a edge 2-coloring **only if a 2-vertex coloring is given**, **we do not have that**
  - The conversion only works on **regular graphs, we do not have that**
  - The recursion schema solves $O(\Delta)$**-edge coloring,** not $(2\Delta - 1)$ **-edge coloring.** For a better result, we need to solve a harder variant **(list coloring)**

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- This problem can be solved in $O(\Delta^5/k^5)$ rounds!
- For $k = \dfrac{\Delta}{\log \Delta}$, this gives an $O(\log^5 \Delta)$ round algorithm!

Open questions: edge coloring

- We can solve $(2\Delta - 1)$-edge coloring in $O(\log^{12} \Delta + \log^* n)$ rounds
  - Can we improve the exponent? We know a faster algorithm, but only for $O(\Delta)$-edge coloring
- Can we solve vertex coloring in subpoly(Δ)?
- Can we prove a non-trivial lower bound for solving (2Δ - 1)-edge coloring?
  - Can we show that it cannot be solved in o(log Δ) + O(log* n)?

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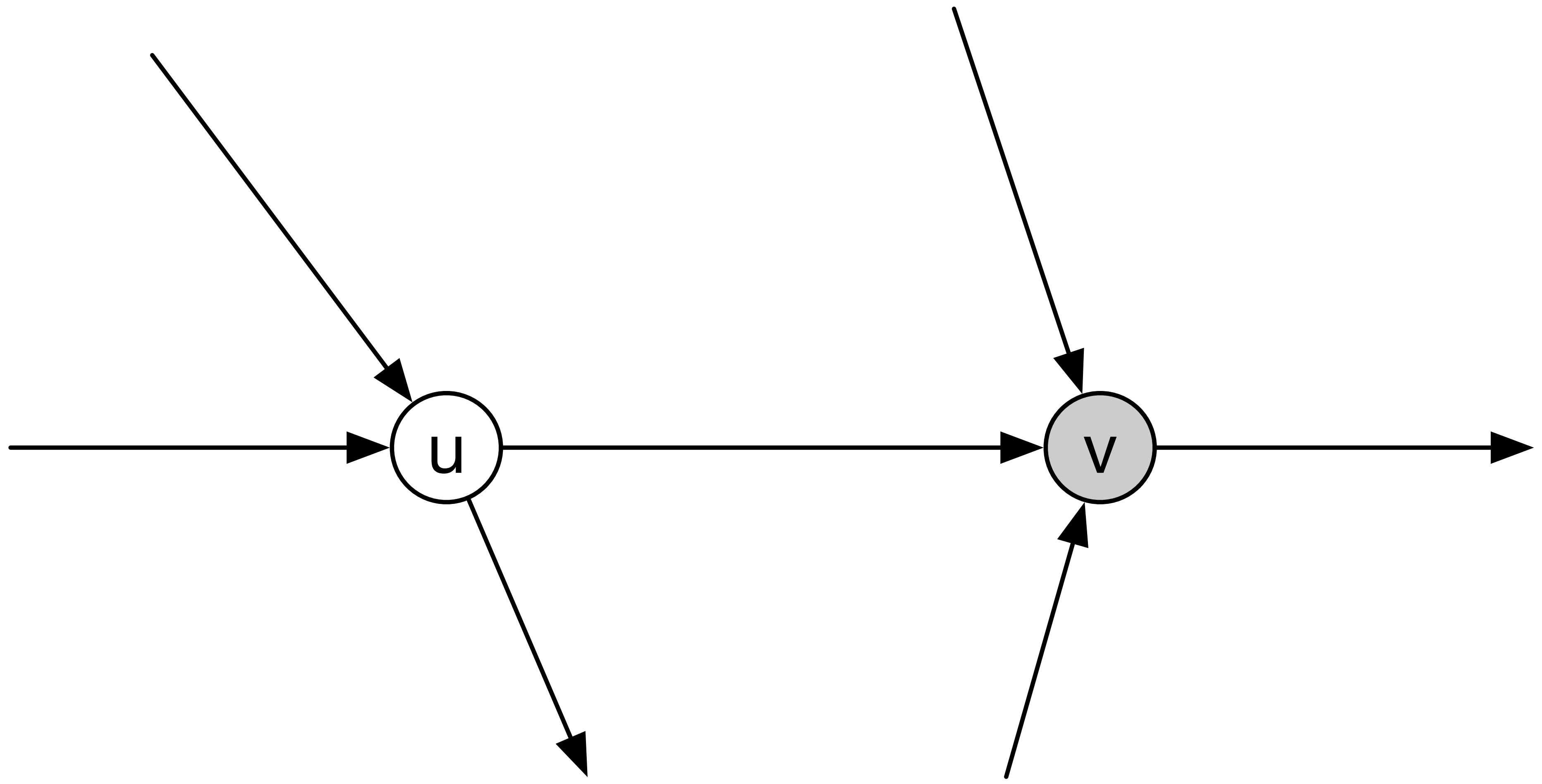
  [Brandt, Keller, Rybicki, Suomela, Uitto 2021]
- Can we do better?



Thank you!