On Pareto Optimality in Social Distance Games

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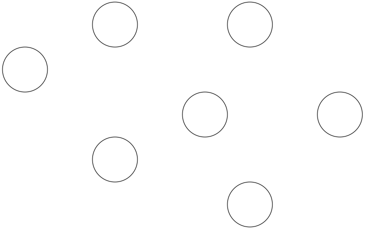
Goal and Motivation

- Study the social networks from the point of view of cooperative game theory.
- *Social Distance Games (SDGs)*: a model of interaction on social networks capturing the idea that in social networks agents prefer to maintain ties with agents who are close to them. Introduced by Brânzei and Larson.
- Study *Pareto Stability* in SDGs.

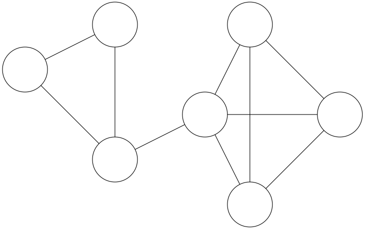
Coalition forming games

- Agents

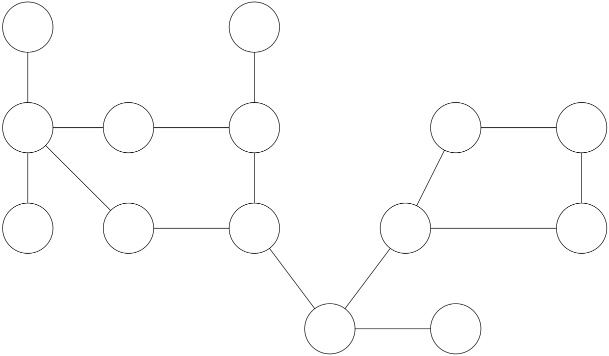


Coalition forming games

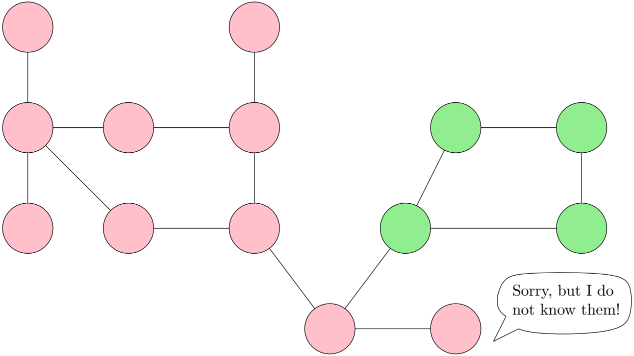
- Agents
- Relations



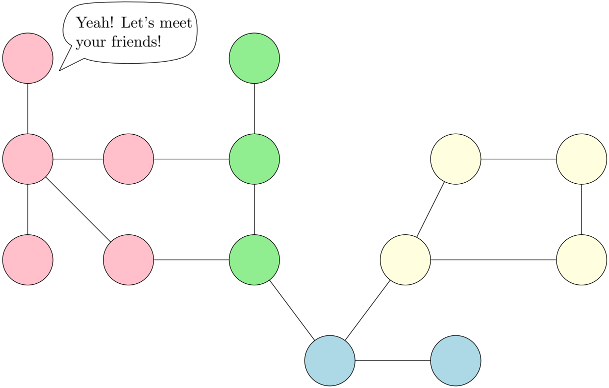
Coalition forming games



Coalition forming games



Coalition forming games



Coalition Forming Games: Utilities' Measures

Let:

- i: an agent
- C: the coalition of the agent i
- E: the set of edges

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Different payoff metrics (happiness of agents):

- Fractional Hedonic Games (FHGs): $\frac{1}{|C|} \sum_{j \in C, (i,j) \in E} w_{i,j}$
- **Social Distance Games (SDGs):** $\frac{1}{|C|} \sum_{x_j \in C \setminus \{x_i\}} \frac{1}{dist(x_i, x_j)}$

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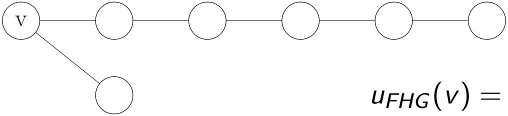
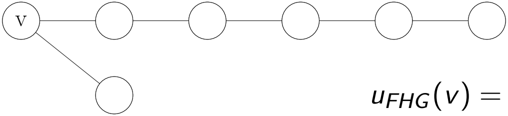
- **Social Distance Games (SDGs):** $\frac{1}{|C|}\sum_{x_j\in C\setminus\{x_i\}} \frac{1}{dist(x_i,x_j)}$
- Additively Separable Hedonic Games: $\sum_{j\in C,(i,j)\in E} w_{i,j}$
- Modified Fractional Hedonic Games: $\frac{1}{|C|-1}\sum_{j\in C,(i,j)\in E} w_{i,j}$
- ...

FHGs vs SDGs: Fractional Hedonic Games



$$u_{FHG}(v) = \frac{2}{7}$$

FHGs vs SDGs: Fractional Hedonic Games



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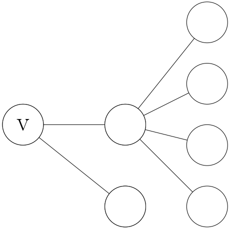

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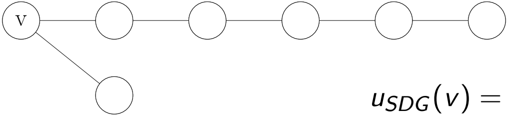
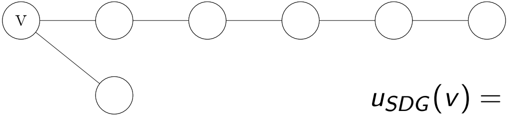


$$u_{SDG}(v) = \frac{2+1/2+1/3+1/4+1/5}{7} \approx 0.46$$

FHGs vs SDGs: Social Distance Games



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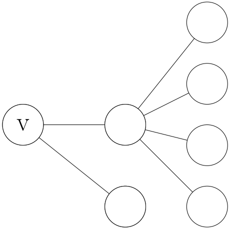
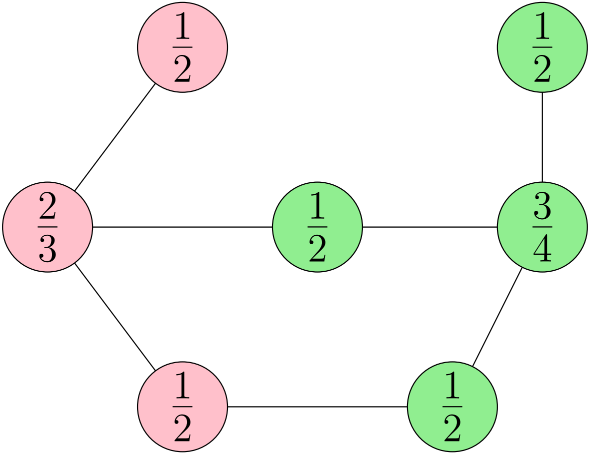


$$u_{SDG}(v) = \frac{2+4/2}{7} \approx 0.57$$

Social welfare



$$SW = \sum_{v \in V} u_{SDG}(v) = 5 * \frac{1}{2} + \frac{2}{3} + \frac{3}{4} \approx 3.92$$

Notions of Stability

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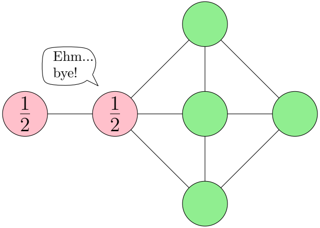
- *Nash Stability*: each agent is selfish and at each step one agent can deviate.
- *Core Stability*: a group of agents can deviate simultaneously to form a new coalition.
- **Pareto Stability**: all agents can deviate simultaneously, forming any new set of coalitions, but nobody should get a lower utility.

Nash



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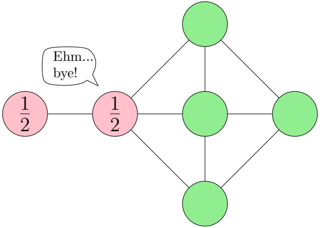

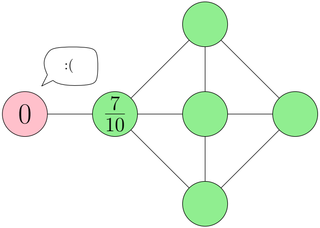

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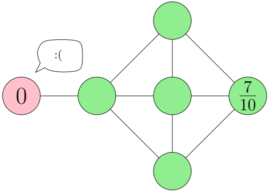
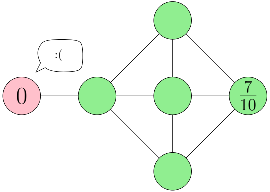
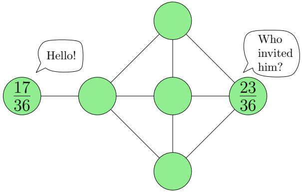
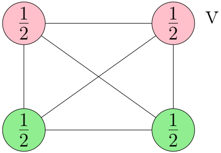


Nash



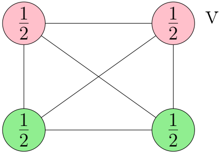


Pareto



$u_{SDG}(v) = \frac{1}{2}$
$SW = 2$

Pareto



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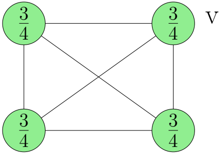
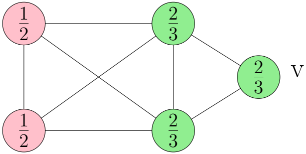


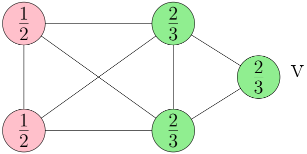
$$u_{SDG}(v) = \frac{3}{4}$$
$$SW = 3$$

Pareto



$u_{SDG}(v) = 2/3$
$SW = 3$

Pareto



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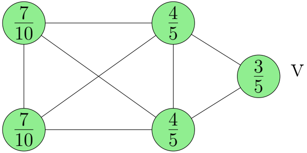
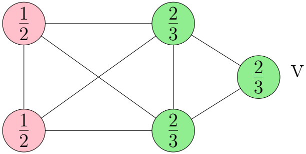


$u_{SDG}(v) = 3/5$
$SW = 3.6$

Pareto



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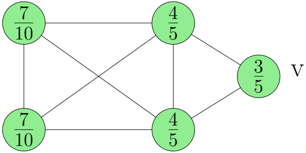

$u_{SDG}(v) = 3/5$
$SW = 3.6$

Not a valid deviation! $3/5 < 2/3$

Price of Pareto Optimality (PPO)

A game is *Pareto Optimal* if there is no other outcome in which all players are at least as well off and some players are strictly better off.

- A group of players can not deviate if the utility of some other player decreases.
- Every social welfare maximizing outcome is Pareto Optimal, thus Price of Stability is 1.

Price of Pareto Optimality (PPO)

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Ratio between the social welfare of the best solution, and the social welfare of the worst Pareto stable solution.

Let:

- PO = set of all Pareto stable partitions
- P^* = the partition maximizing the social welfare

$$PPO(G) = max_{P \in PO} \frac{SW(P^*)}{SW(P)}$$

PPO in SDGs

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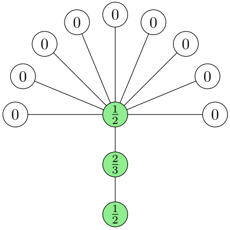
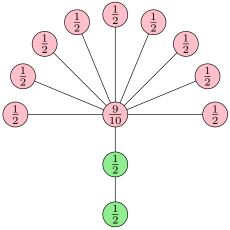
PPO in SDGs is $\Theta(n)$



$SW = \Theta(1)$



$SW^* = \Theta(n)$

Undirected	Unweighted	Weighted
General	$\Theta(n)$	$\Theta(nW)$
Δ-bounded	$\Theta(\Delta)$	$\Omega(\Delta W)$, $O(min\{nW, \Delta W^2\})$

Directed	Unweighted	Weighted
General	$\Theta(n)$	$\Theta(nW)$
$(1,1)$ bounded	$\Theta(\frac{n}{\log n})$	$\Theta(\frac{nW}{W+\log n} + W)$
$(\Delta,1)$ bounded	$\Theta(\frac{n}{\log\log_\Delta n})$	$\Theta(\frac{nW}{\log\log_\Delta n})$

Finding a Good Partition

If we start from any partition and let the agents coordinate to deviate, the PPO can be very high. We can solve the problem by fixing the initial partition!

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The problem of determining an optimal solution for any G is NP-hard, even when each agent has at most 6 neighbors.

Finding a Good Partition

But, we can find 2 types of solution in polynomial time:

- A Pareto Stable partition P such that

$$\frac{SW(P^*)}{SW(P)} \leq 2min(\Delta, \sqrt{n})$$

- A Pareto Unstable partition such that

$$\frac{SW(P^*)}{SW(P)} \leq 2$$

The second solution is *fair*: each agent achieves a utility of at least $\frac{1}{2}$. The partition could be unstable, but Pareto Stability guarantees that in case of deviations no agent can loose utility!

Conclusion

Main results:

- Pareto Stability seems a fair concept: it forbids deviations where agents can loose utility
- Price of Pareto Optimality can be as bad as $\Theta(n)$
- We can force a good outcome by carefully choosing the initial coalitions

Open problems:

- Find a *stable* partition that achieves a constant approximation of the optimum
- Close the gap of the PPO in Weighted Δ-bounded degree graphs