Distributed Edge Coloring in Time Polylogarithmic in Δ

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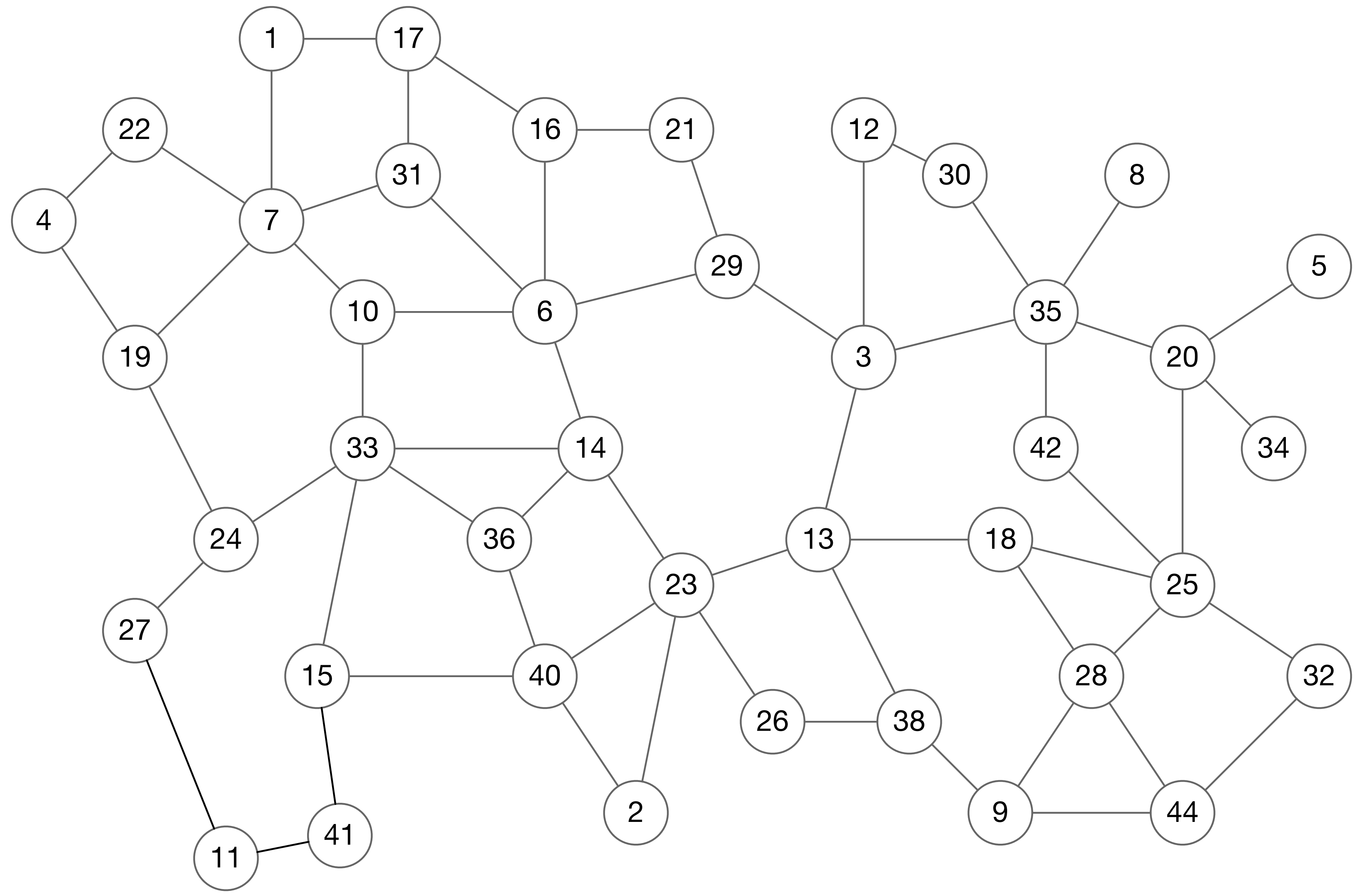
Joint work with: Alkida Balliu, Sebastian Brandt, Fabian Kuhn

LOCAL model

- Undirected simple graph $G = (V, E)$ of n nodes and maximum degree Δ
- Each node has a unique ID
- Synchronous message passing model
- Unbounded computation
- Unbounded bandwidth



CONGEST model

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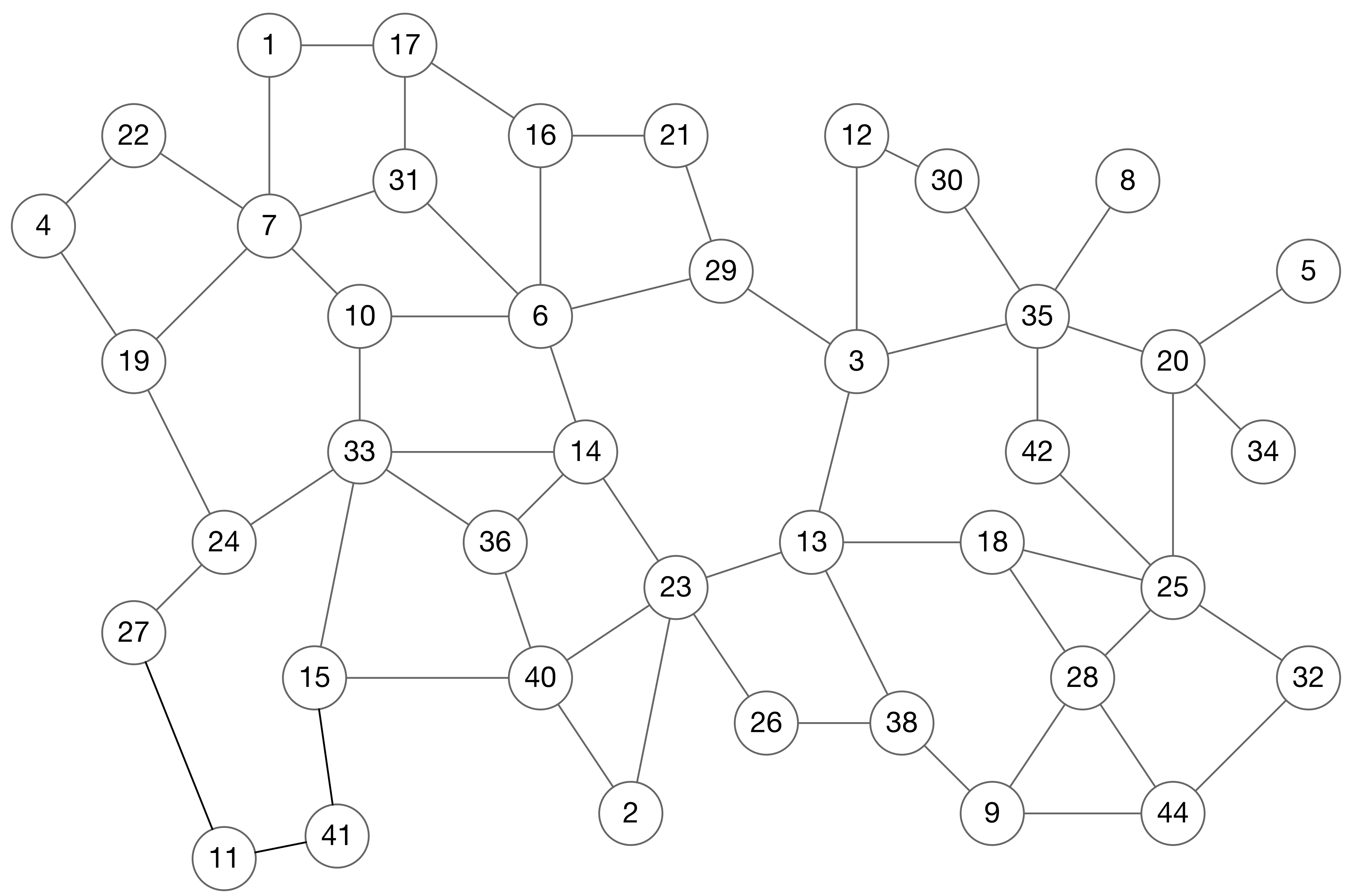
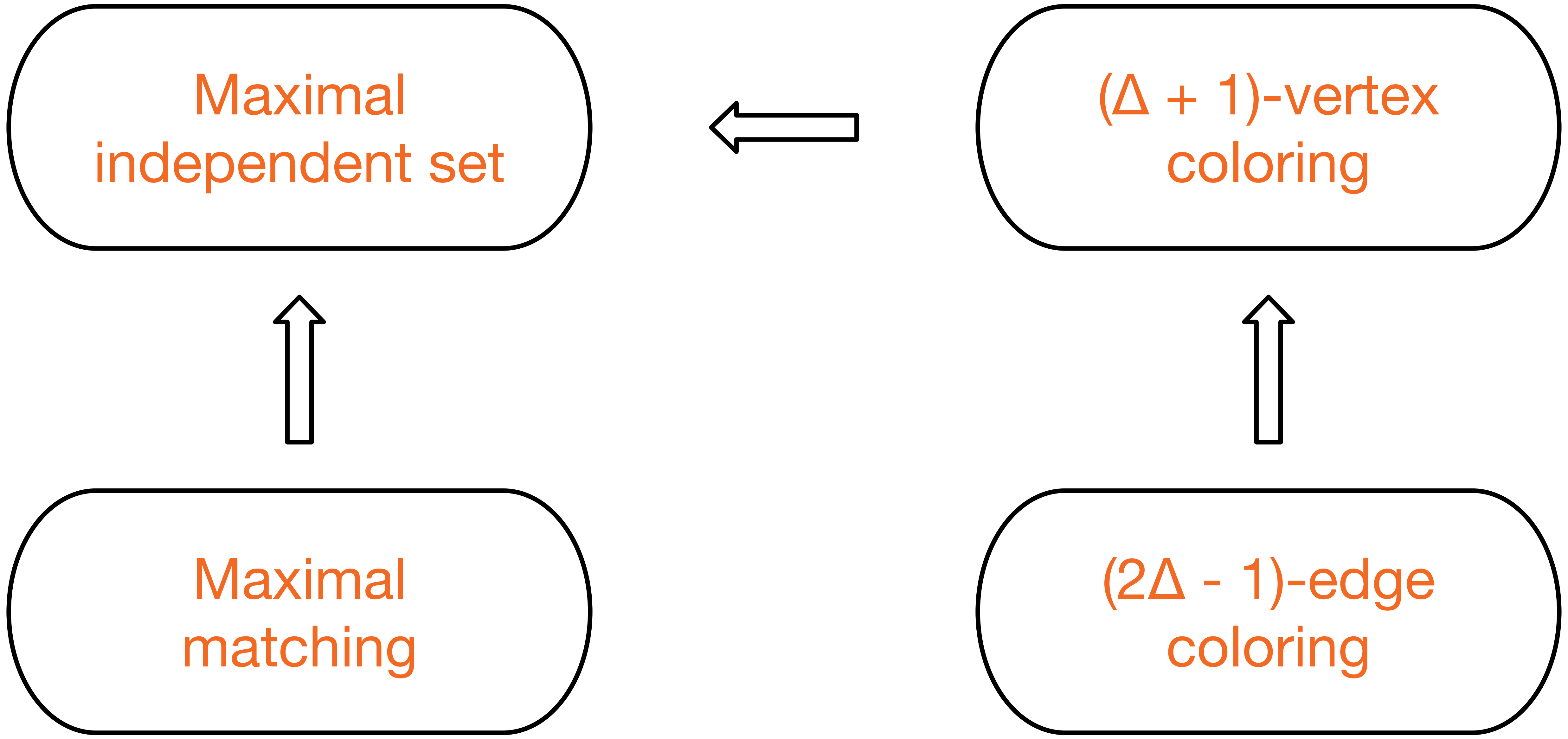
- Each node has a unique ID
- Synchronous message passing model
- Unbounded computation
- O(log n)-bit messages



Four classical problems



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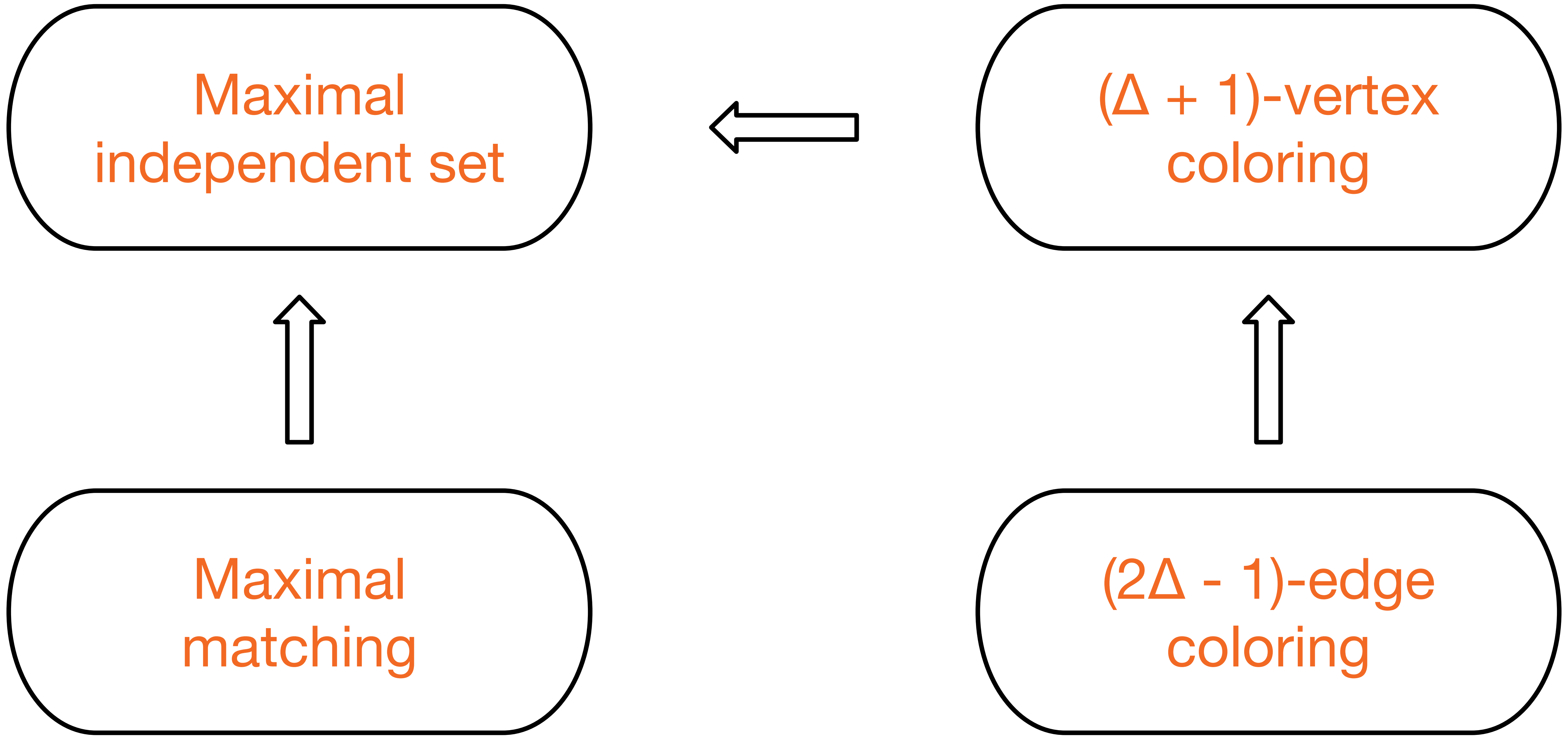

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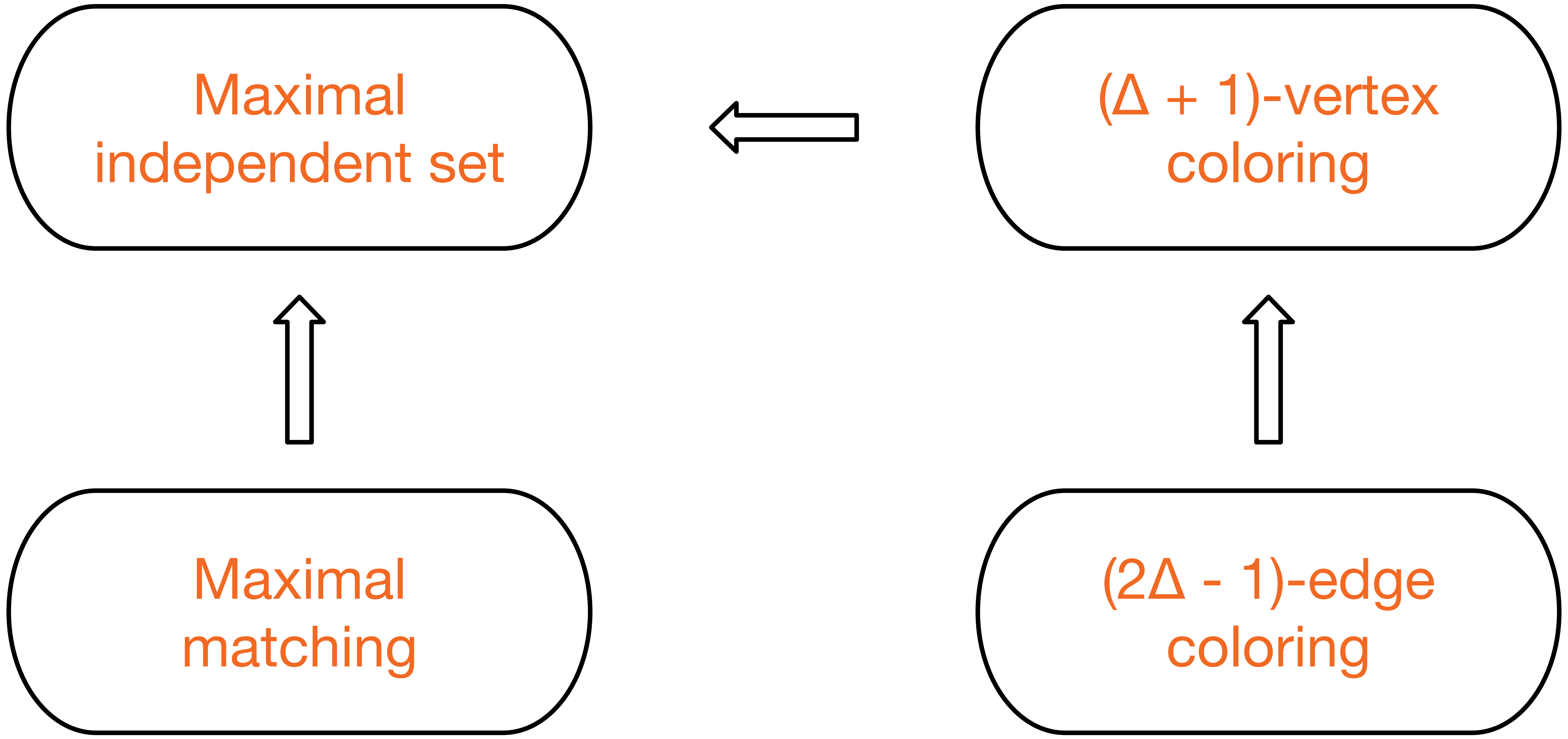

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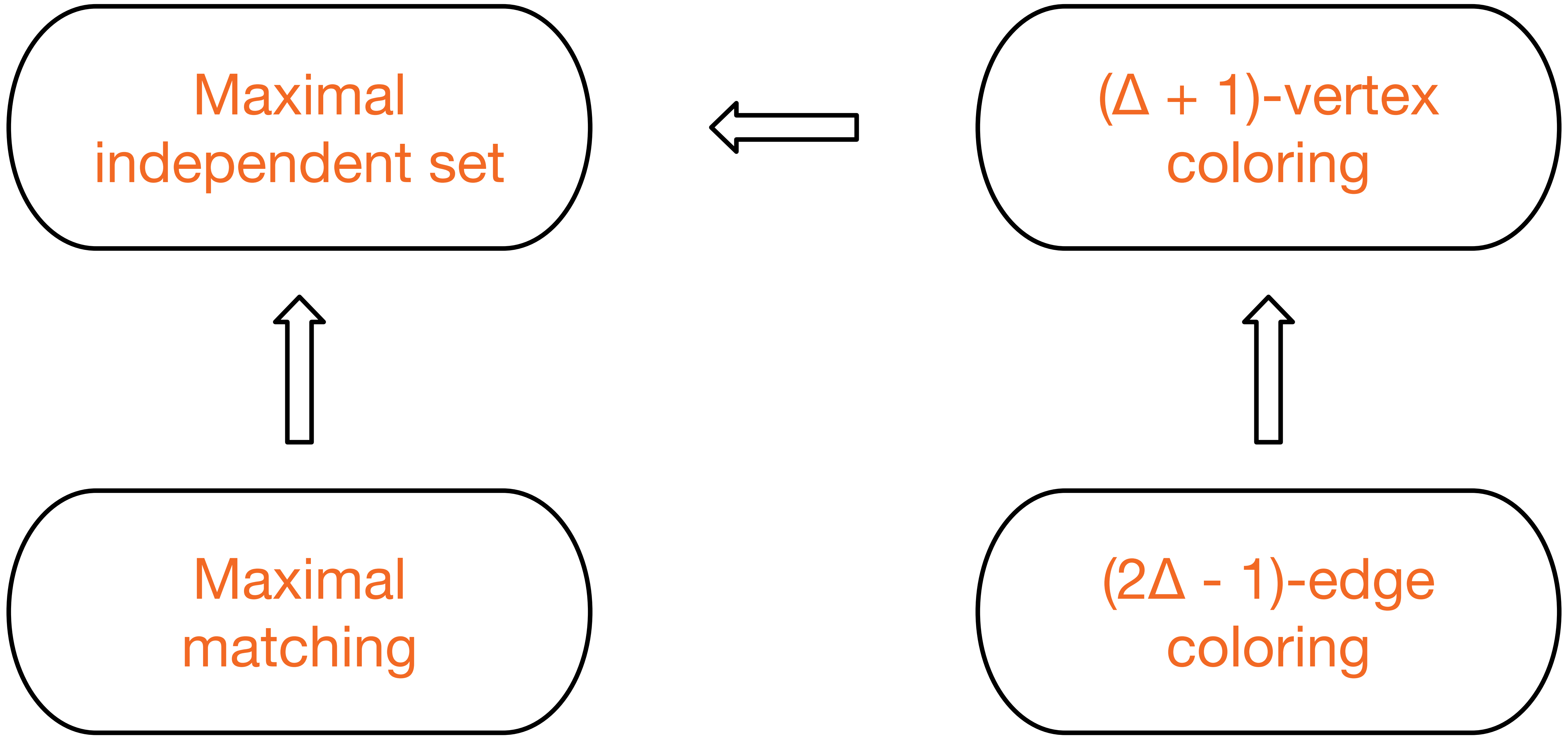

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Big question: $f(\Delta) + O(\log^* n)$

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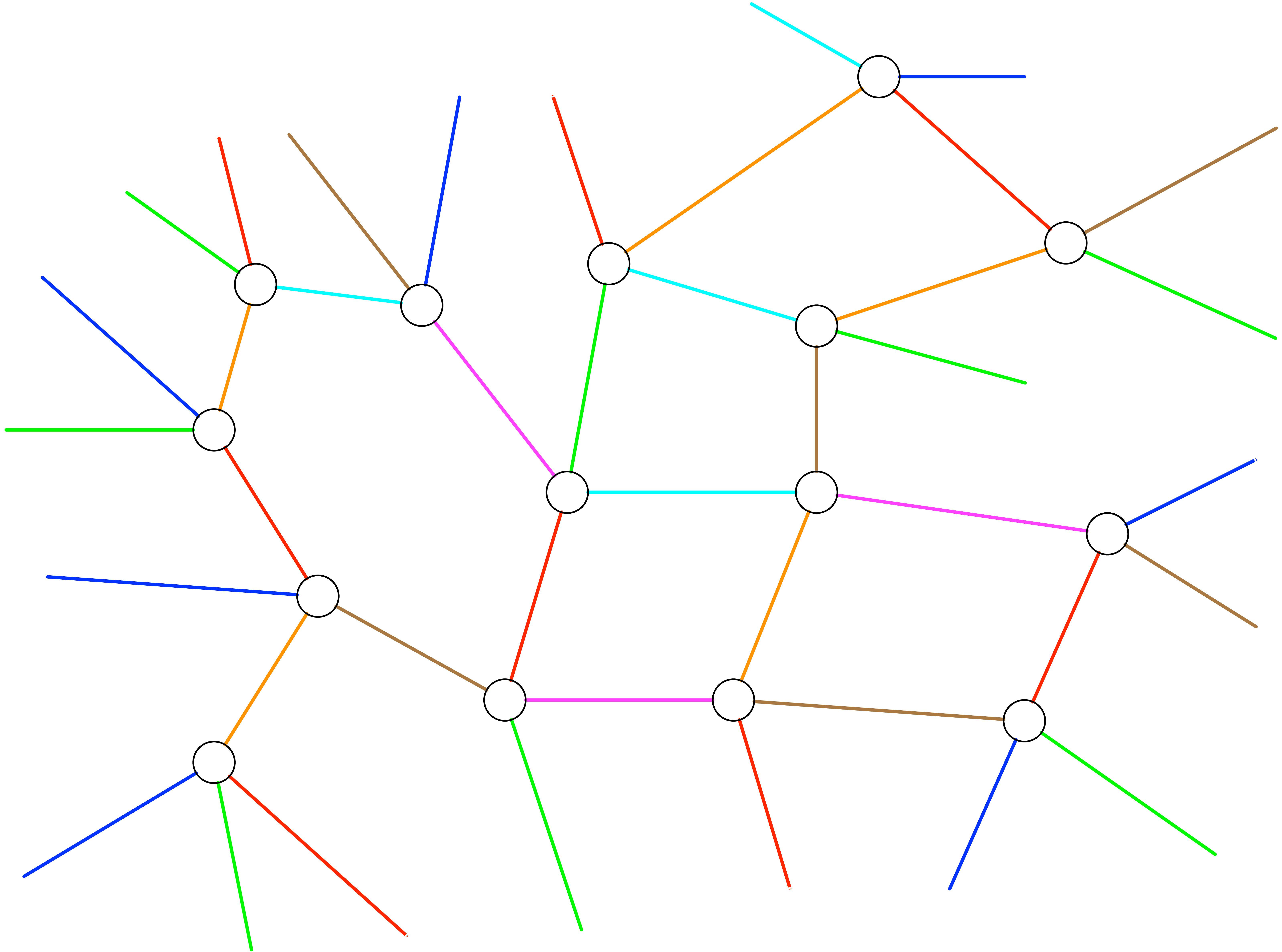

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- **Can we solve $(2\Delta - 1)$-edge coloring in $\operatorname{poly} \log \Delta + O(\log^* n)$ rounds?**

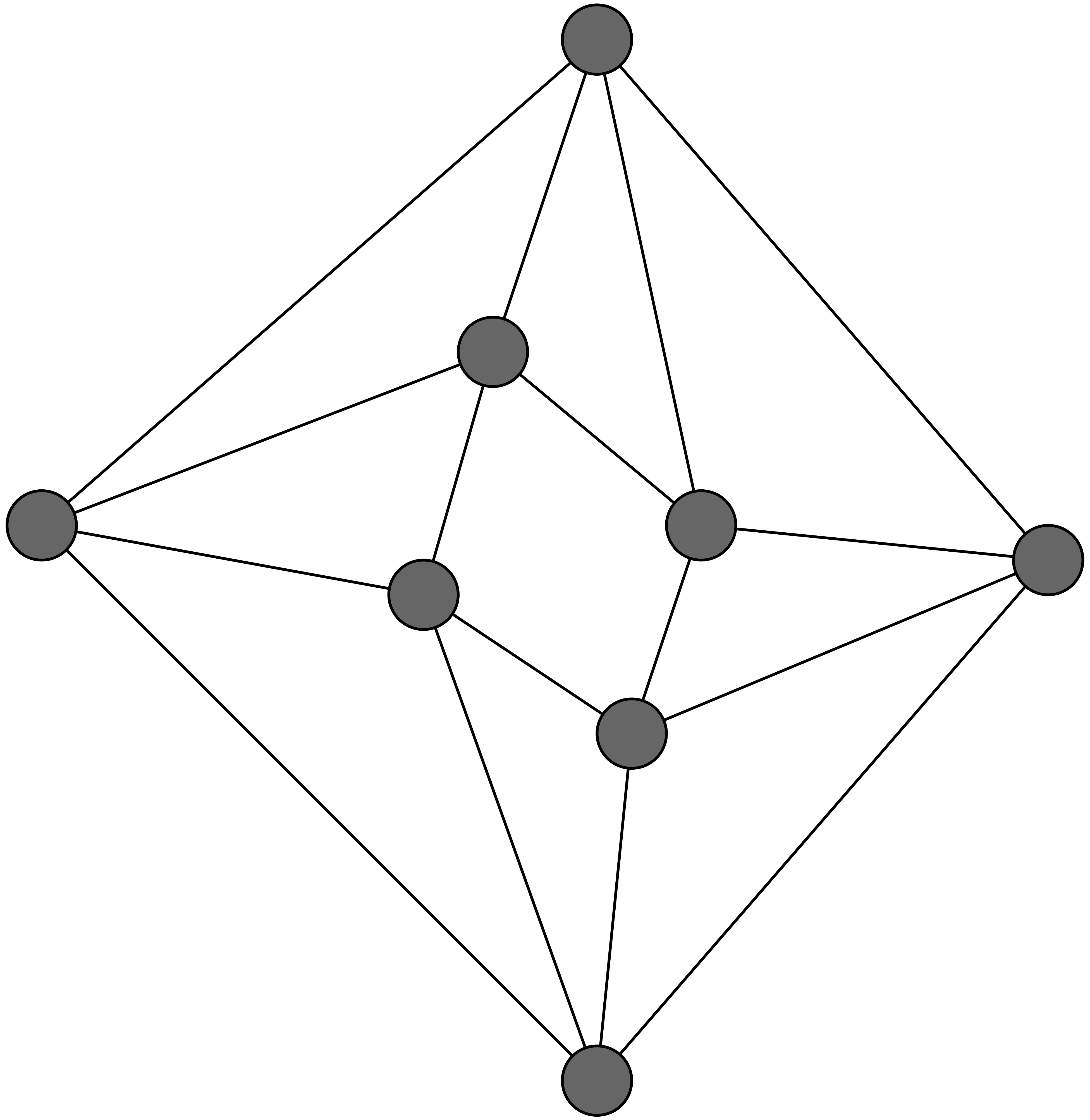
Our results

$(2\Delta - 1)$-Edge Coloring	$O(\text{poly}\log\Delta + \log^* n)$	LOCAL model
$O(\Delta)$-Edge Coloring	$O(\text{poly}\log\Delta + \log^* n)$	CONGEST model

Our results

$(\text{degree}+1)$-List Edge Coloring	$O(\log^7 C \cdot \log^5 \Delta + \log^* n)$	LOCAL model
$(2\Delta-1)$-Edge Coloring	$O(\log^{12} \Delta + \log^* n)$	LOCAL model
$(8+\varepsilon)\Delta$-Edge Coloring	$O\left(\frac{\log^{12} \Delta}{\varepsilon^6} + \log^* n\right)$	CONGEST model

A possible approach



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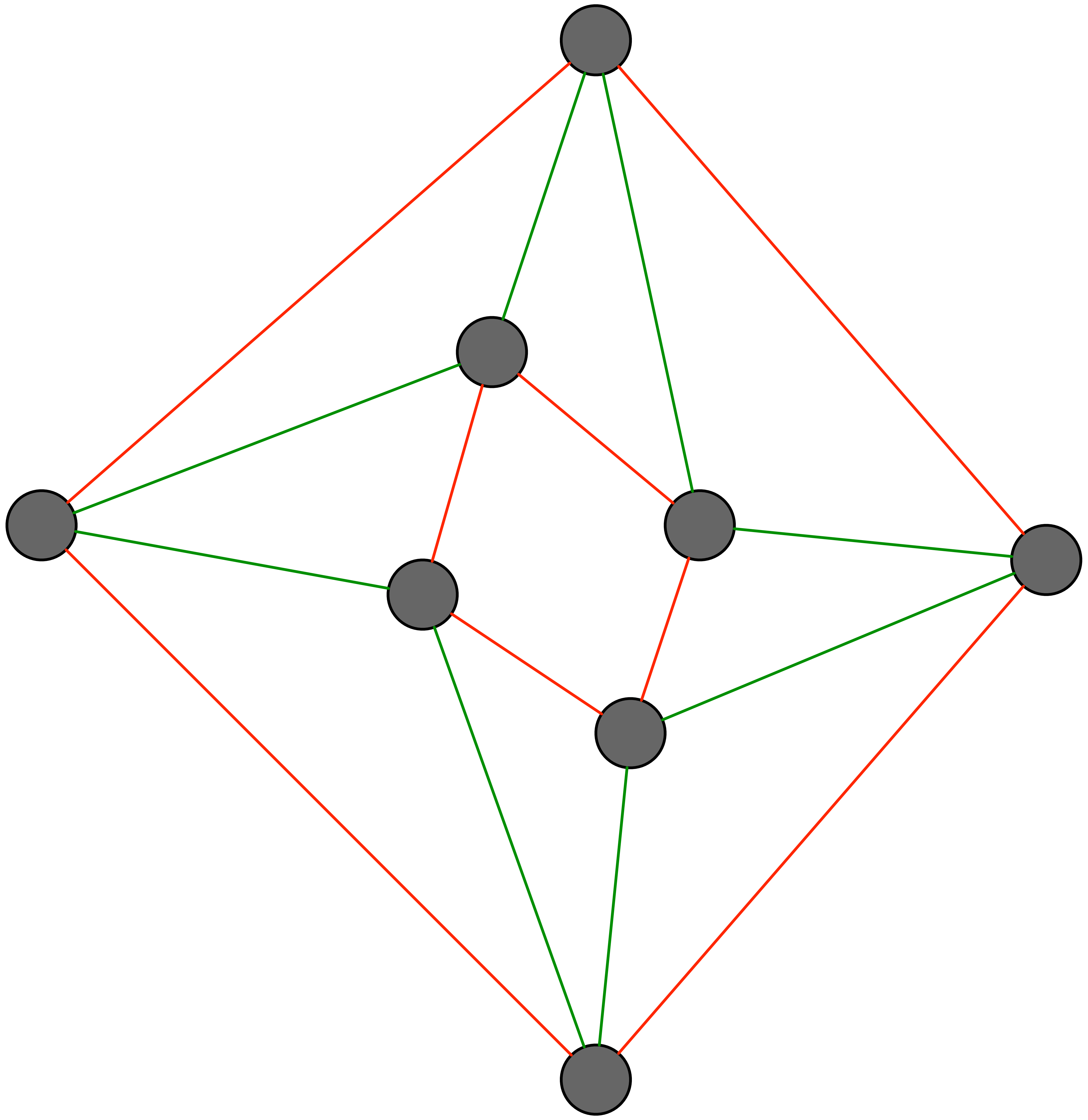

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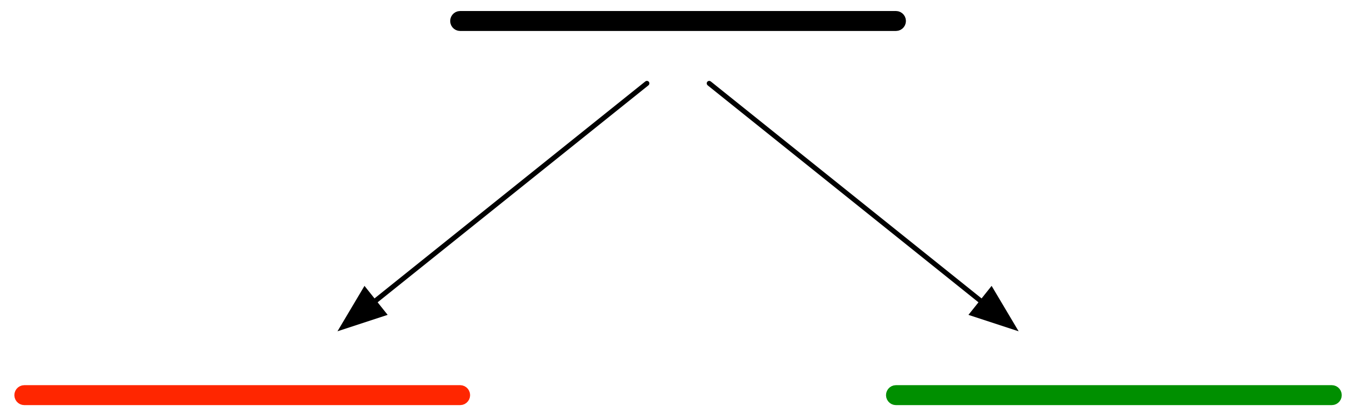

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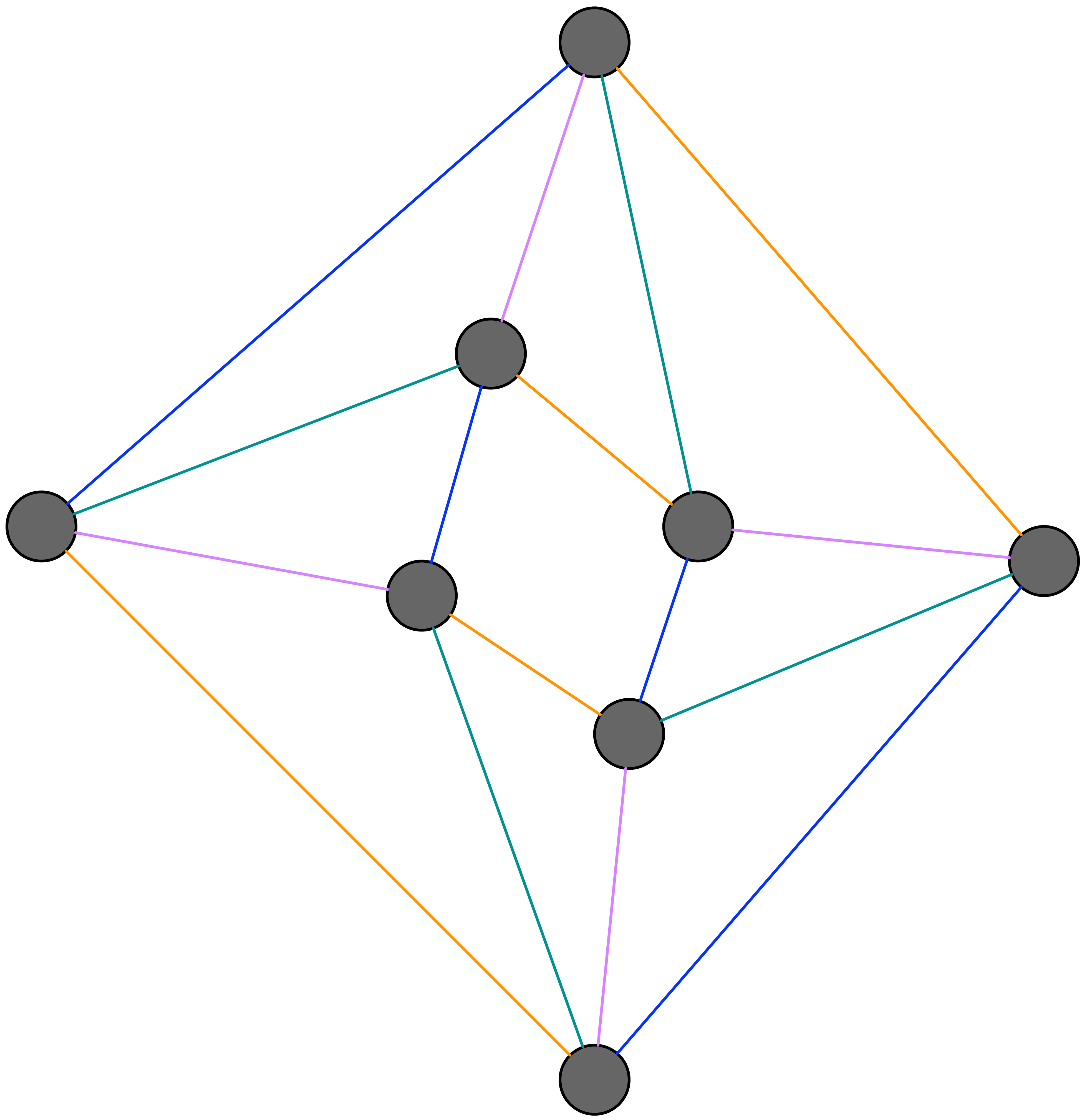

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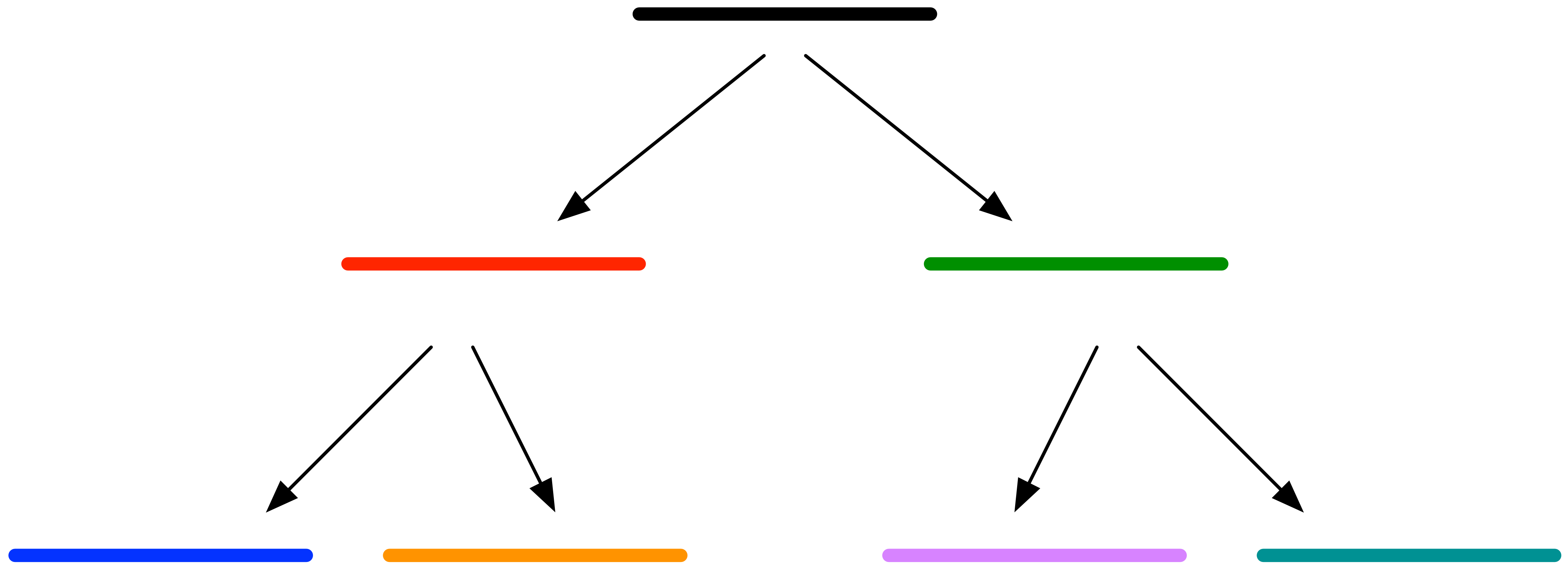

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This requires too much!

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A possible approach: the issue

- **2-coloring** the edges such that each node has **at least one incident edge for each color** requires $\Omega(\log n)$ rounds, but our target runtime is $O(\text{poly} \log \Delta + \log^* n)$

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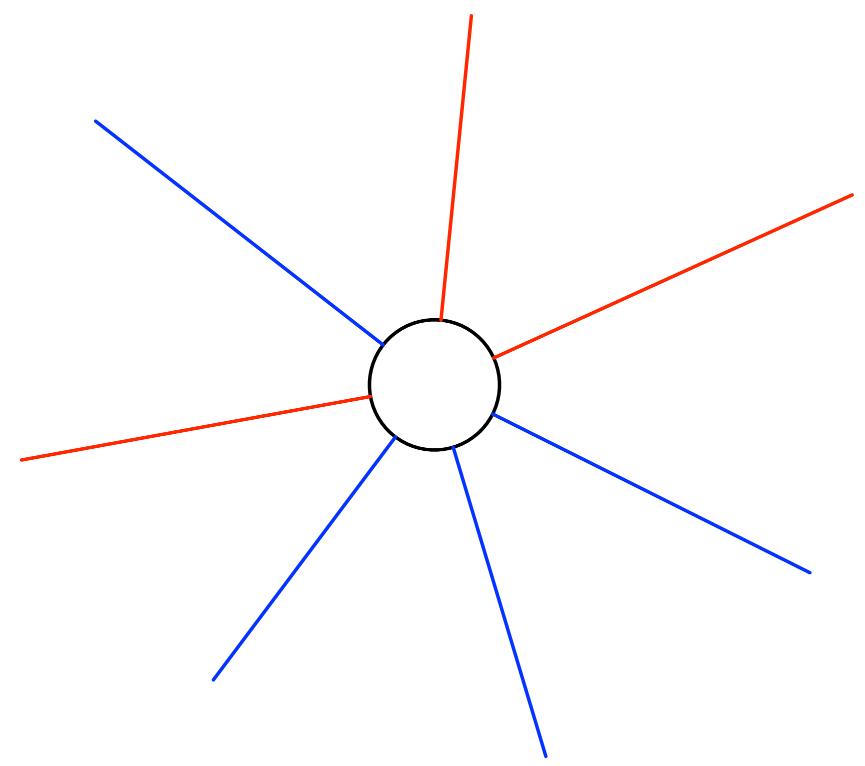

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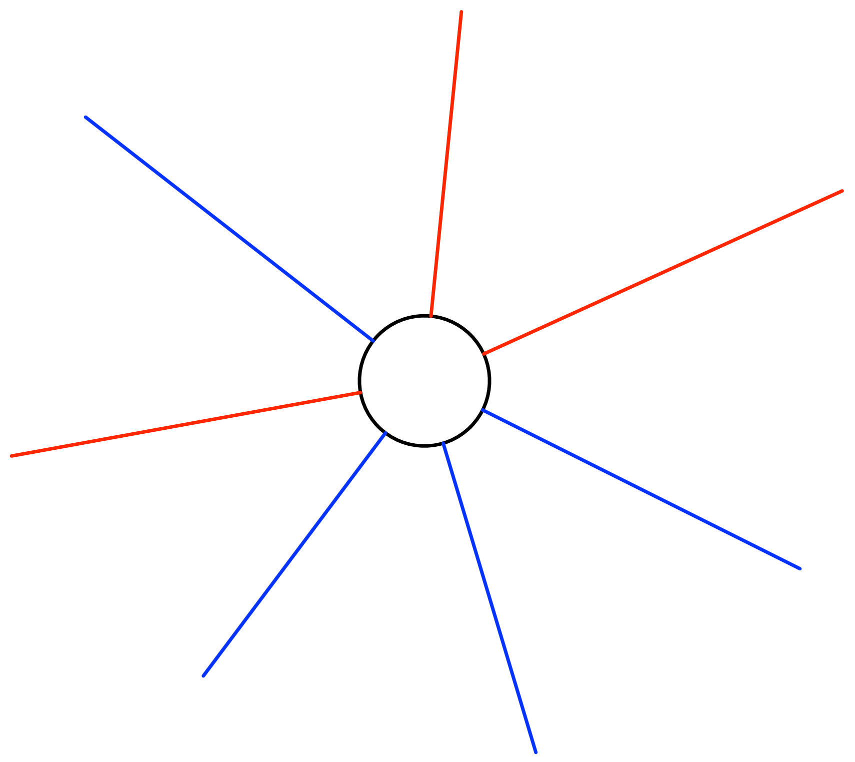

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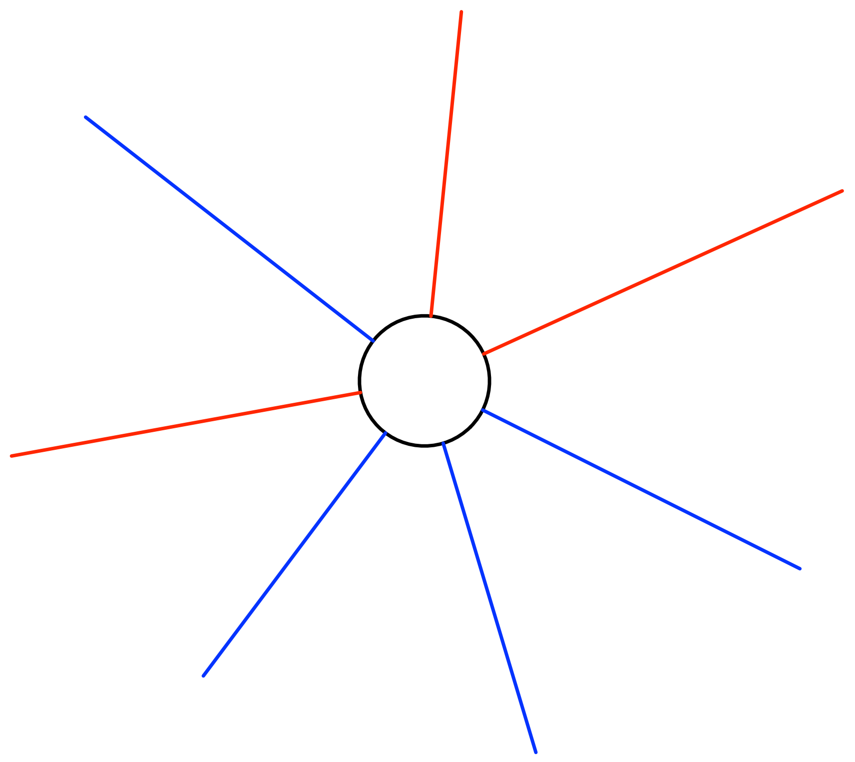

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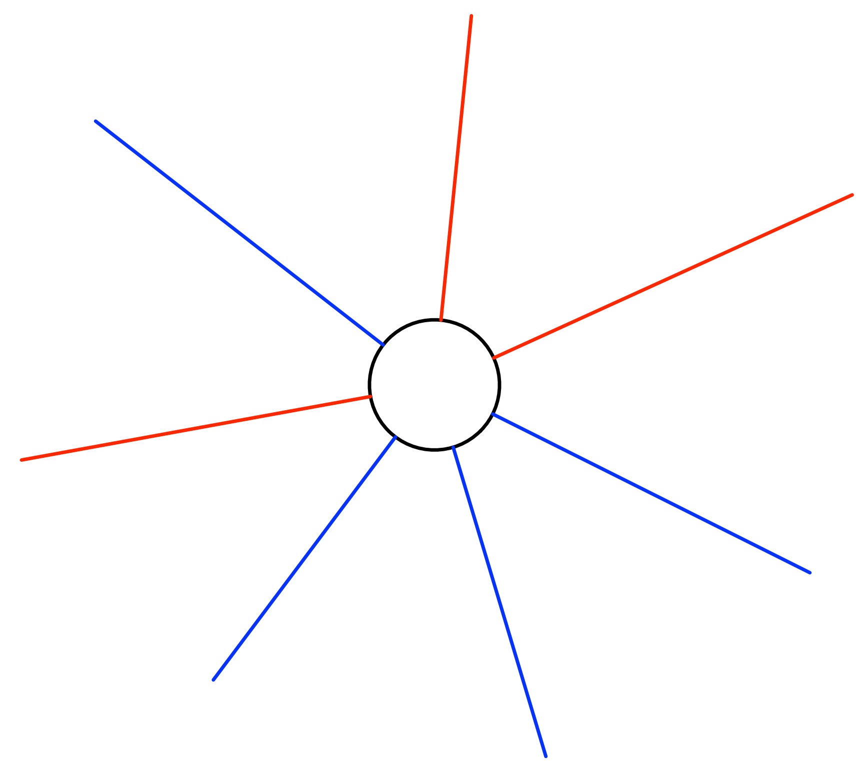

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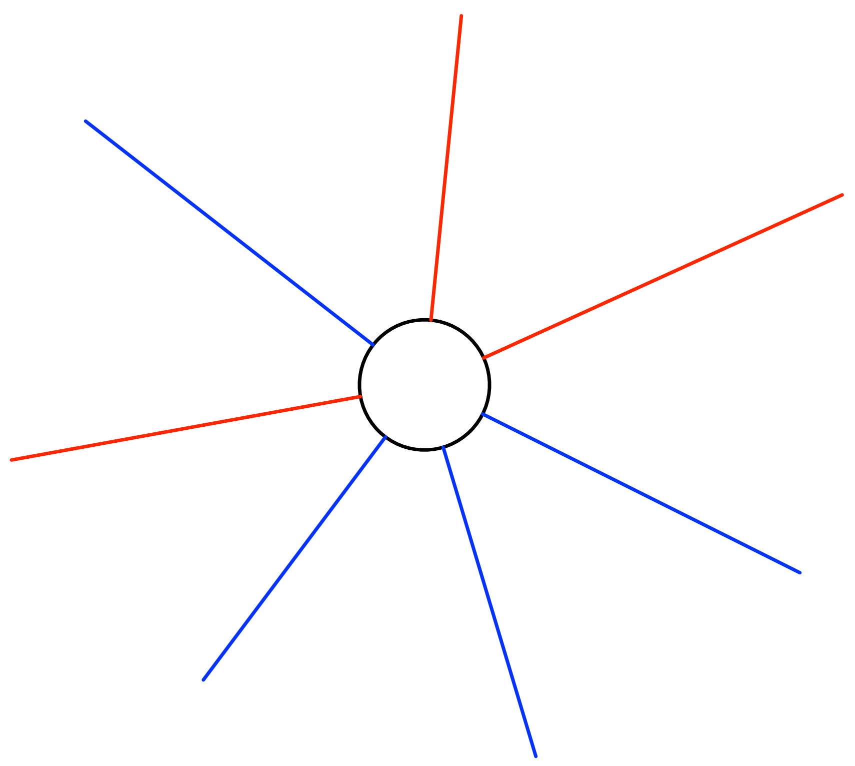

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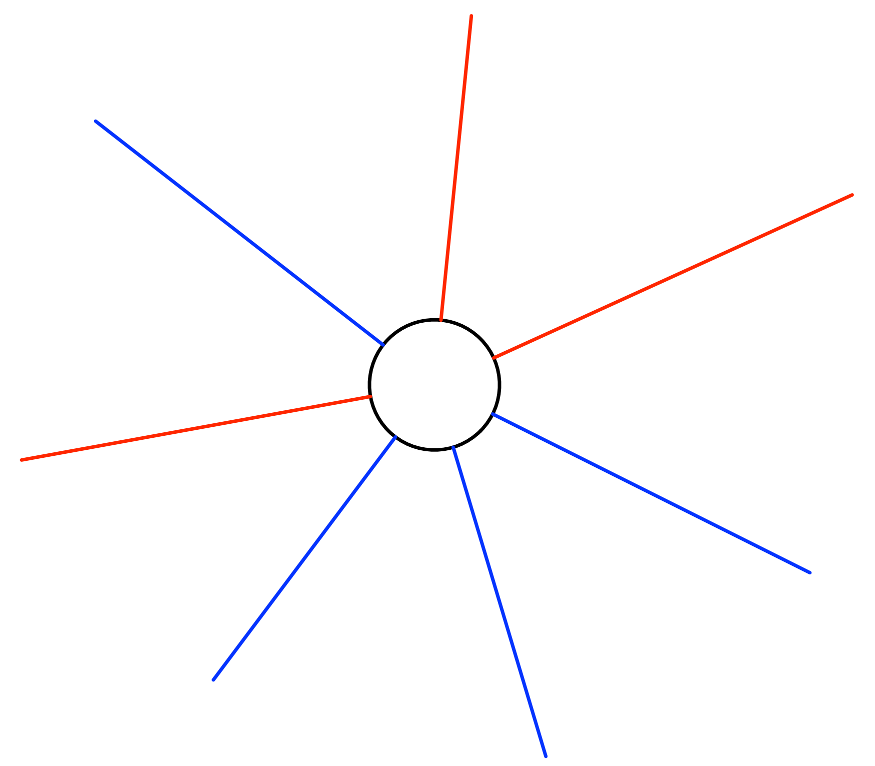

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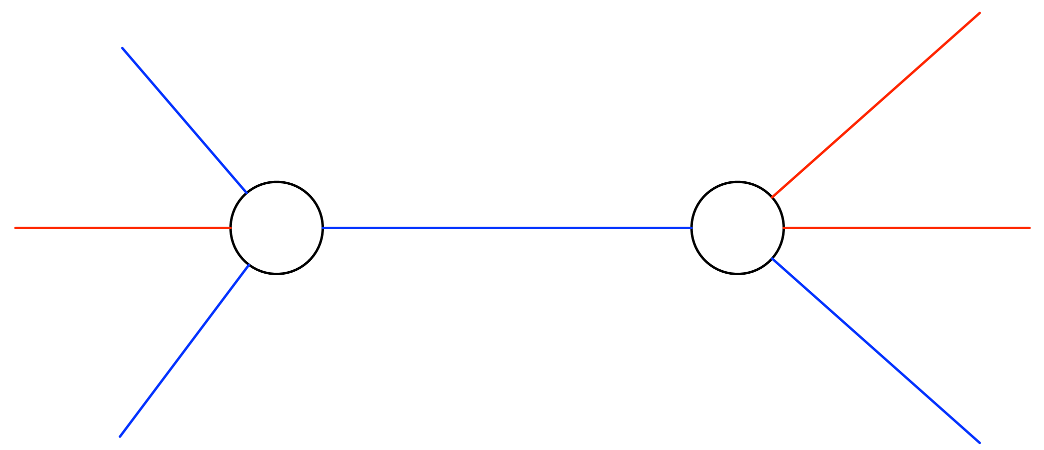

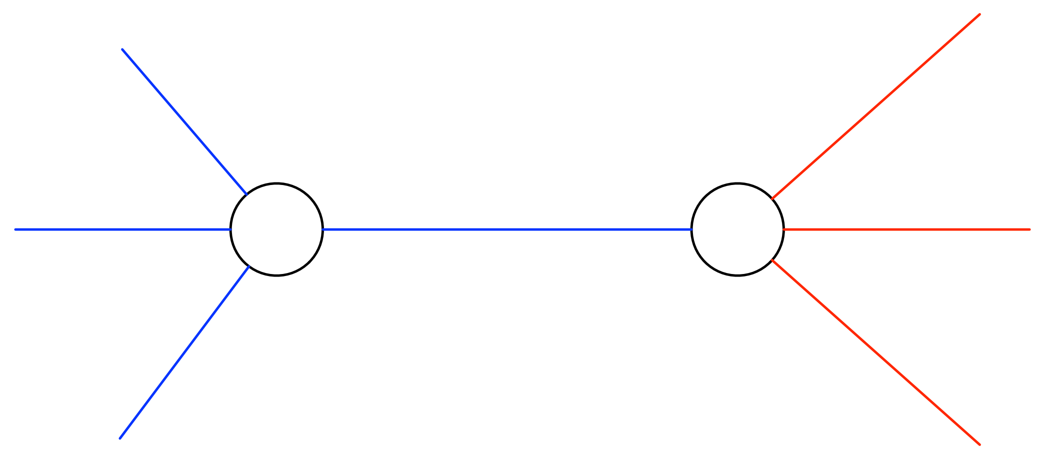

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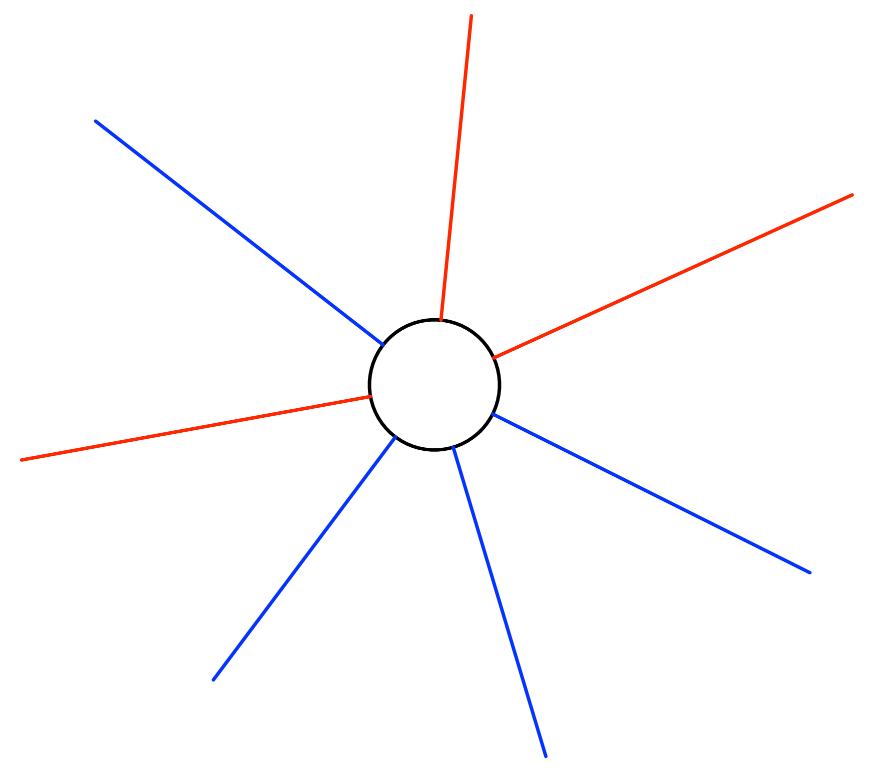

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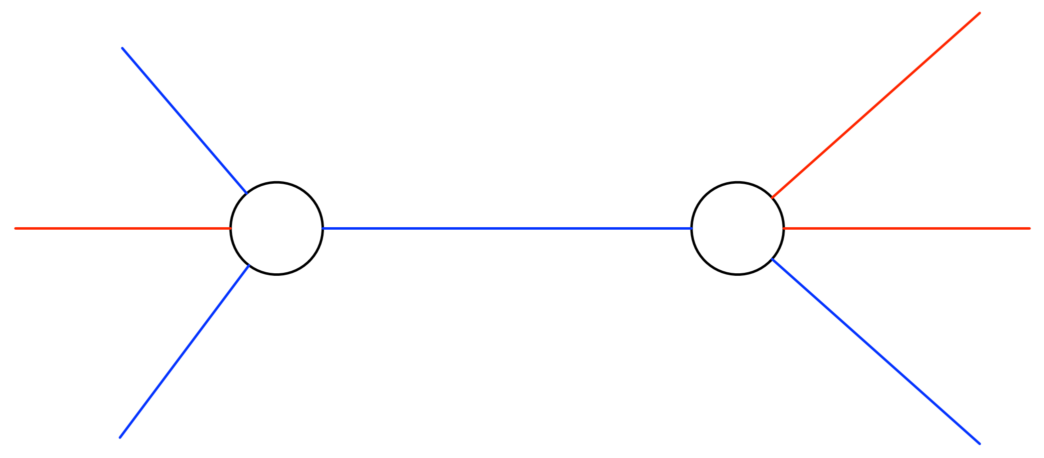

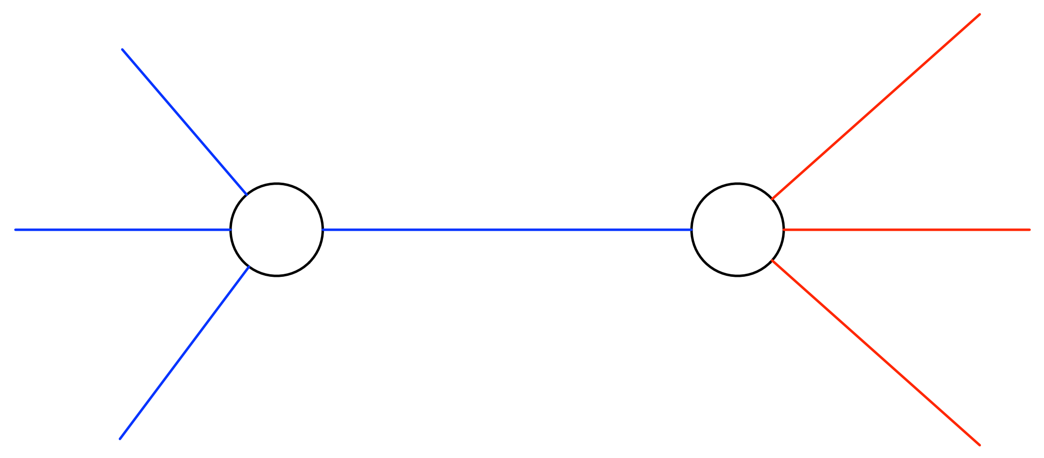

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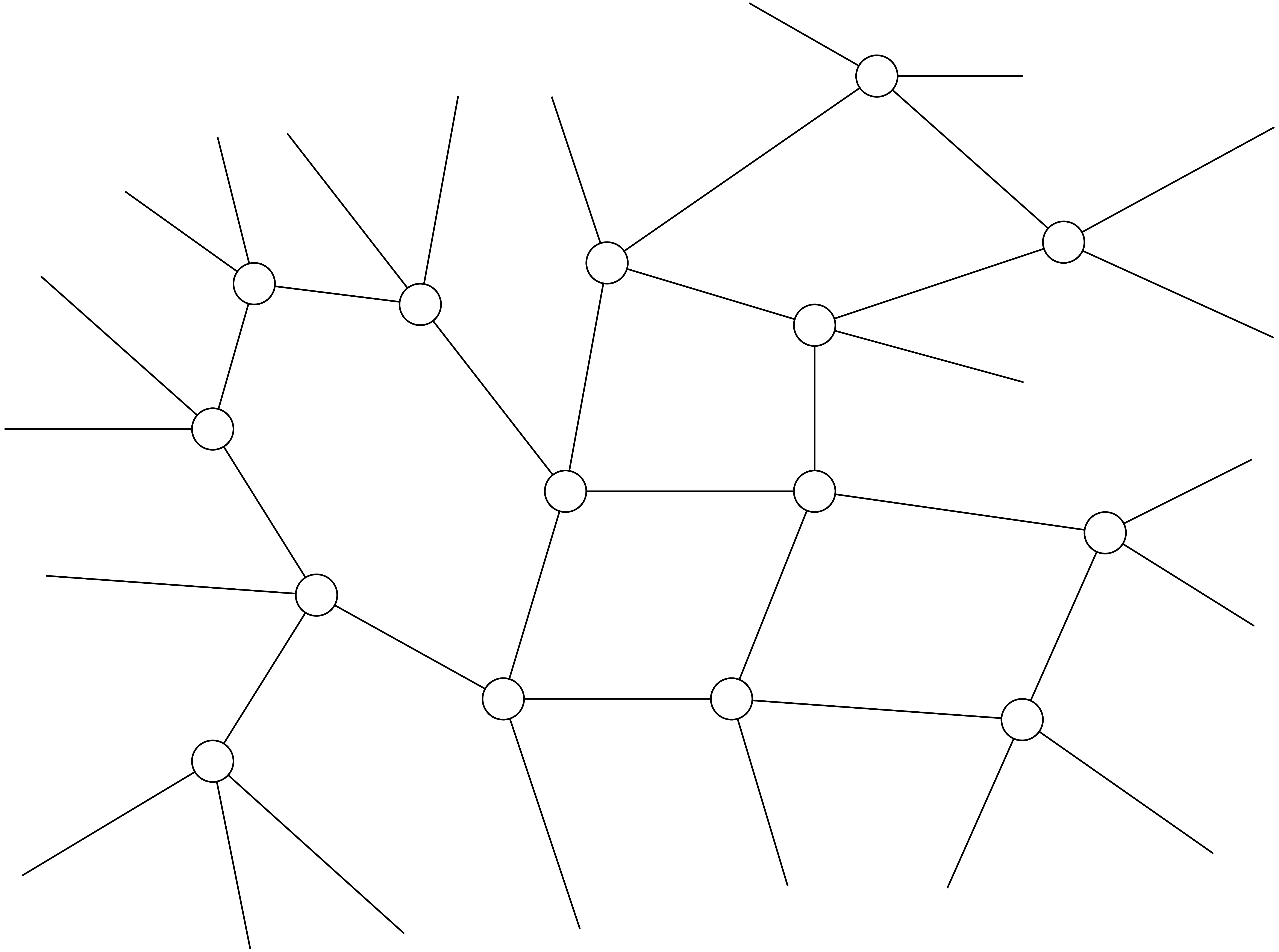

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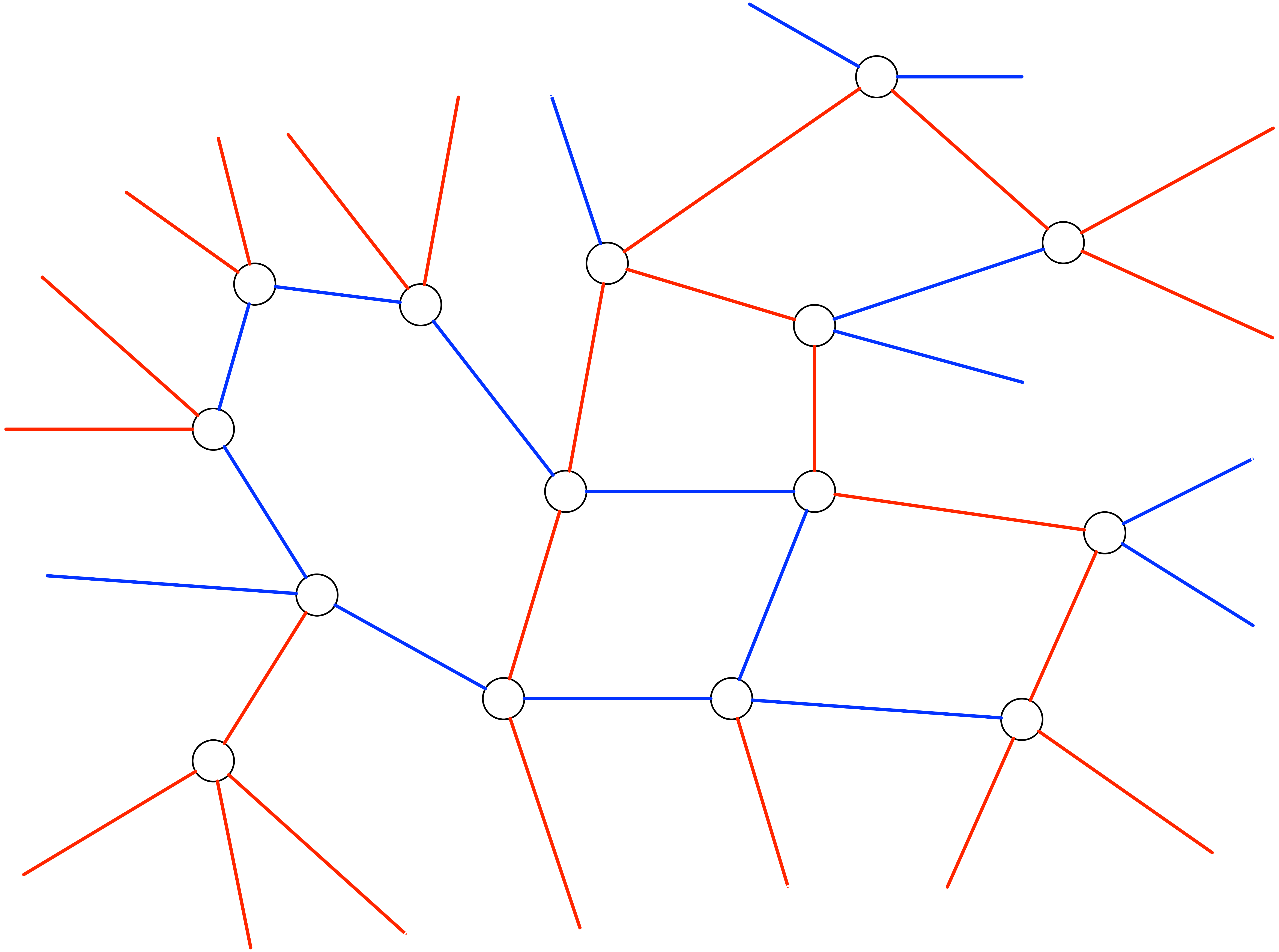

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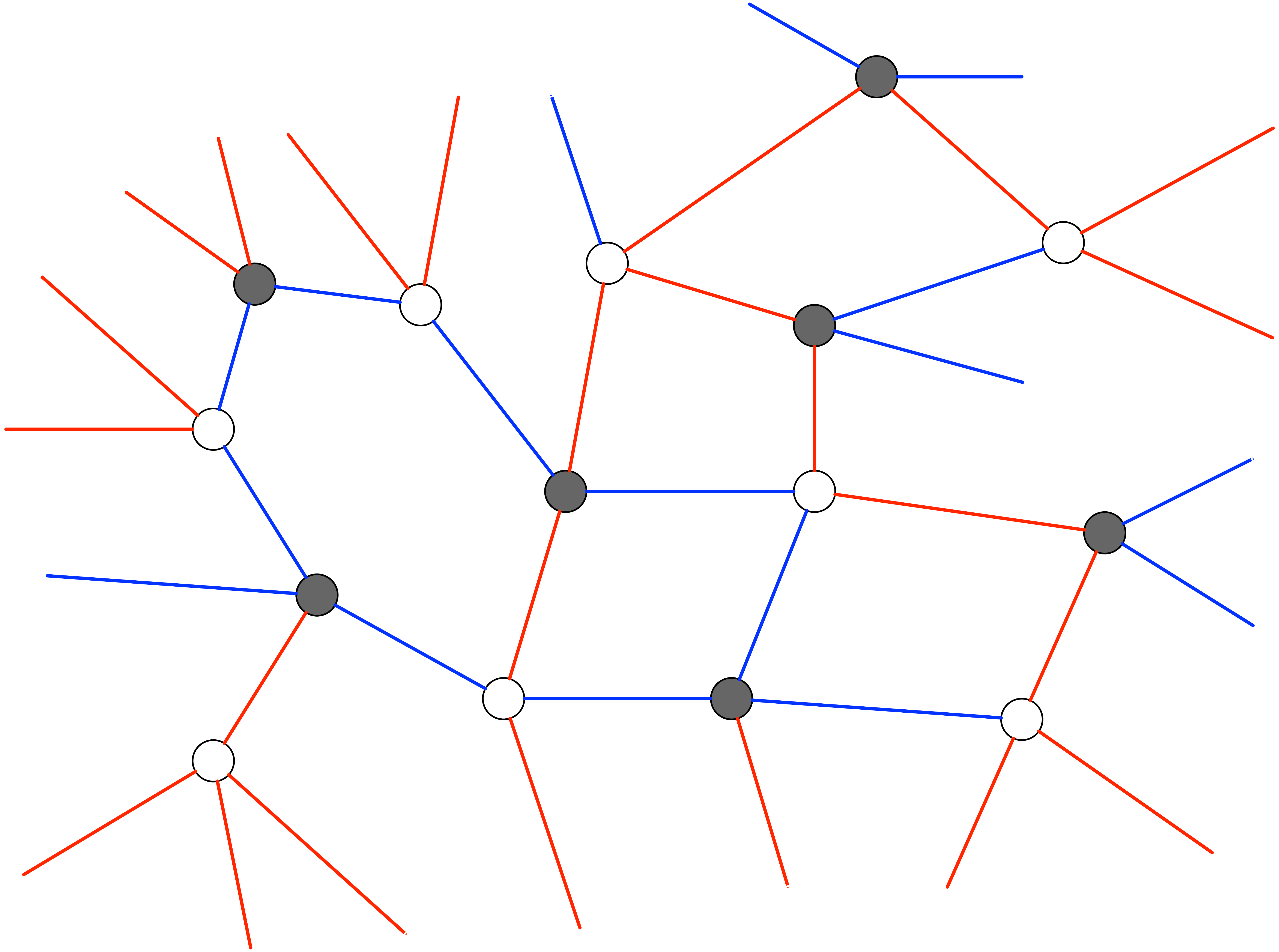

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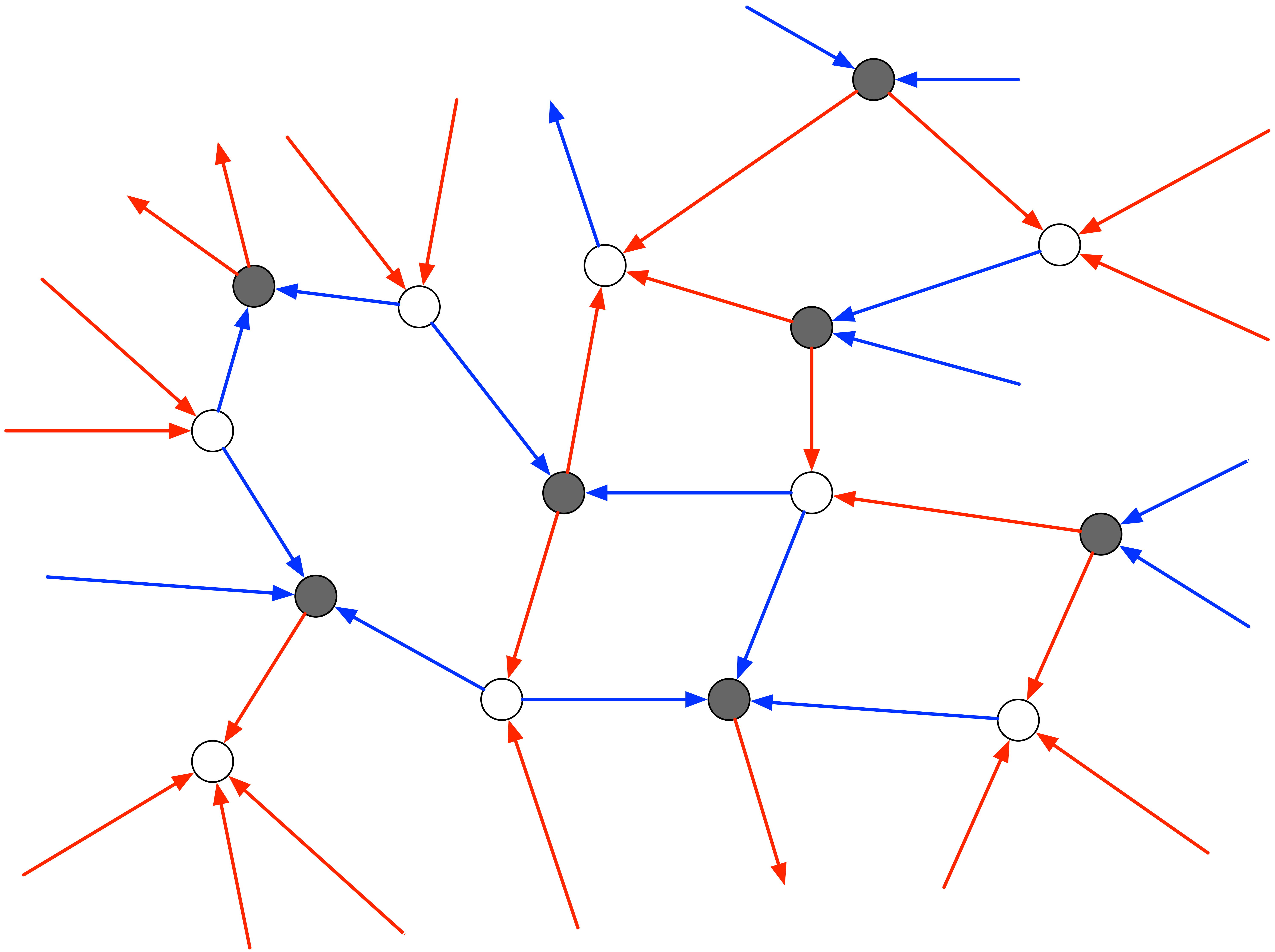

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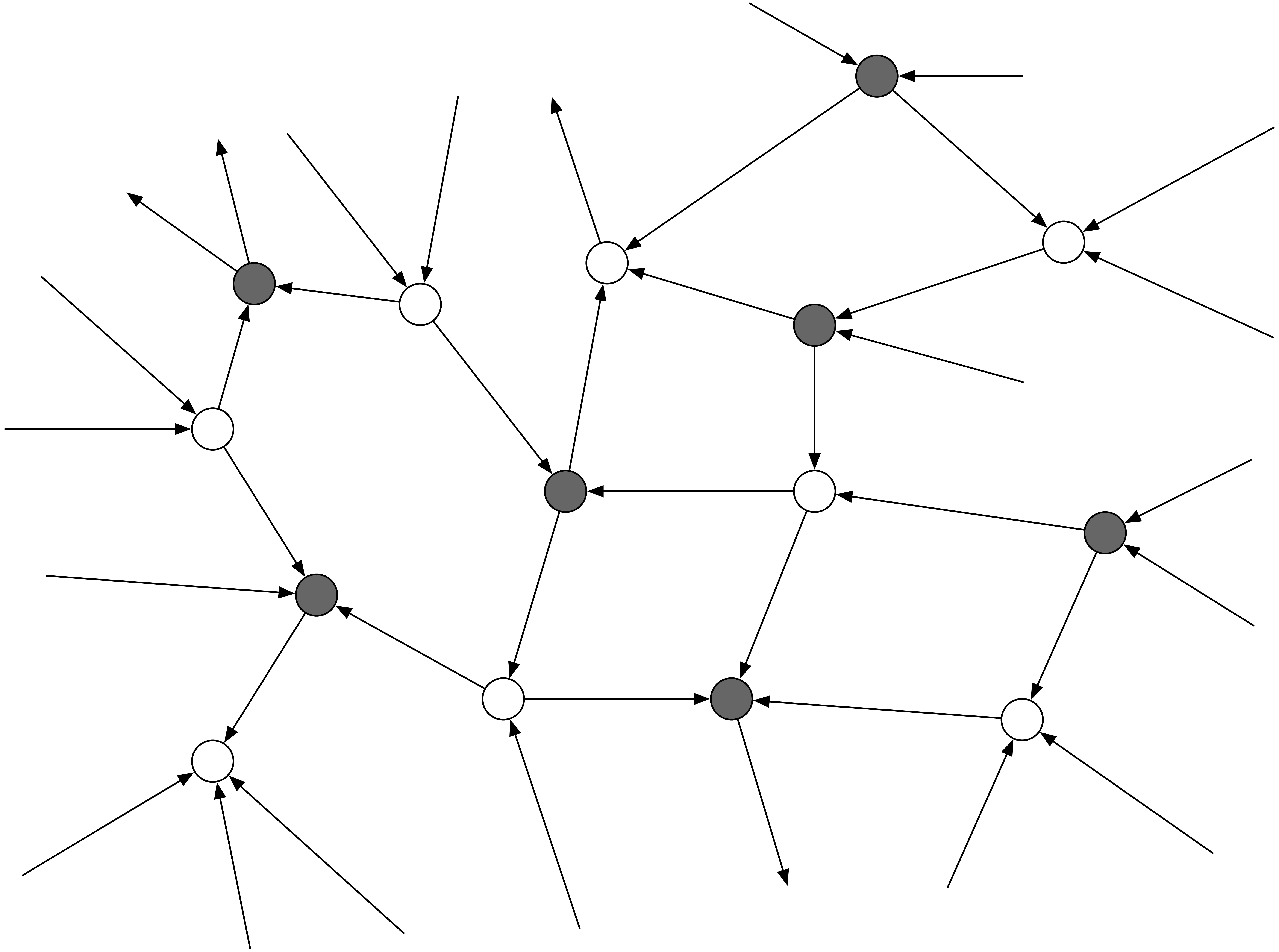

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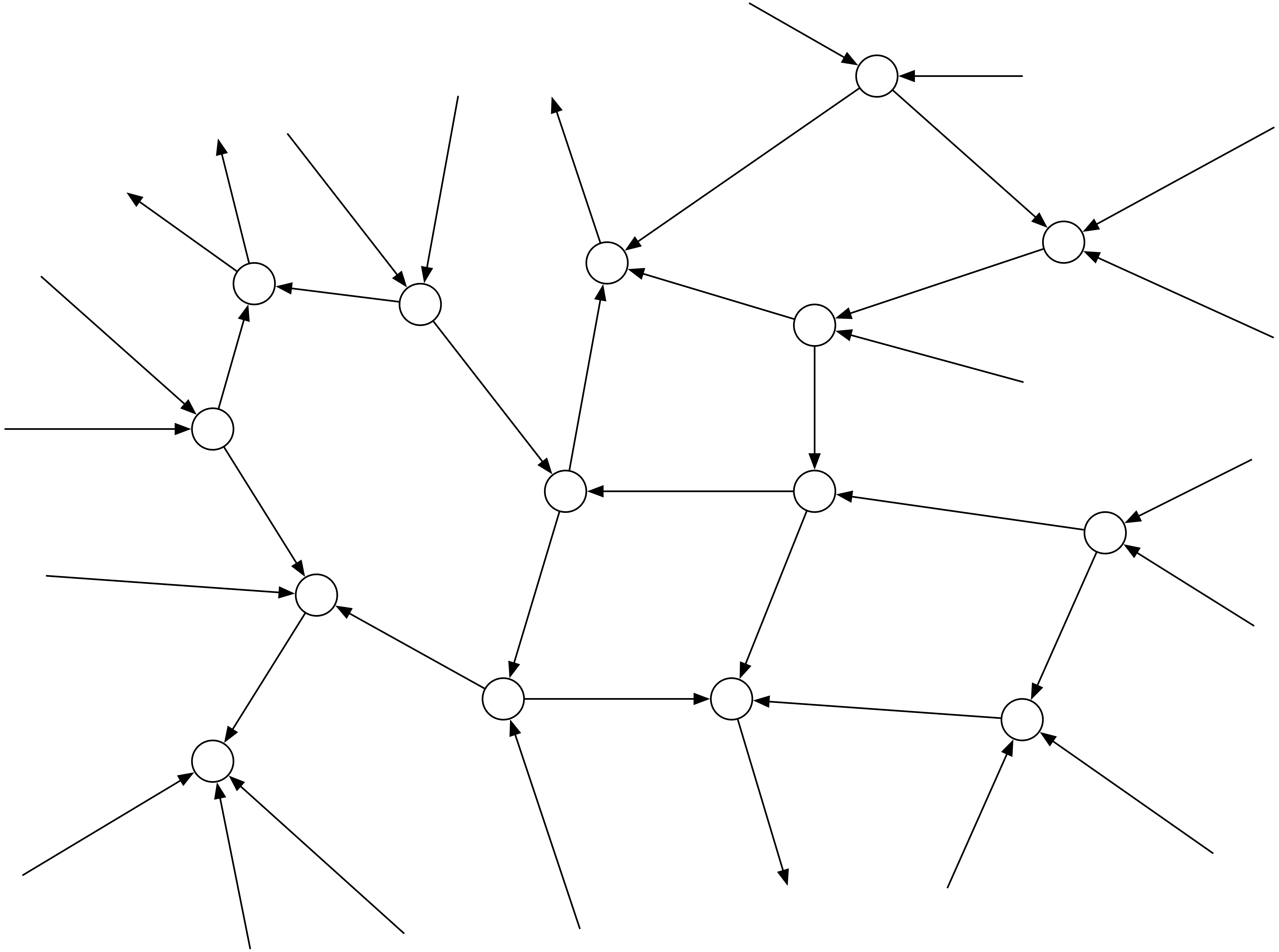
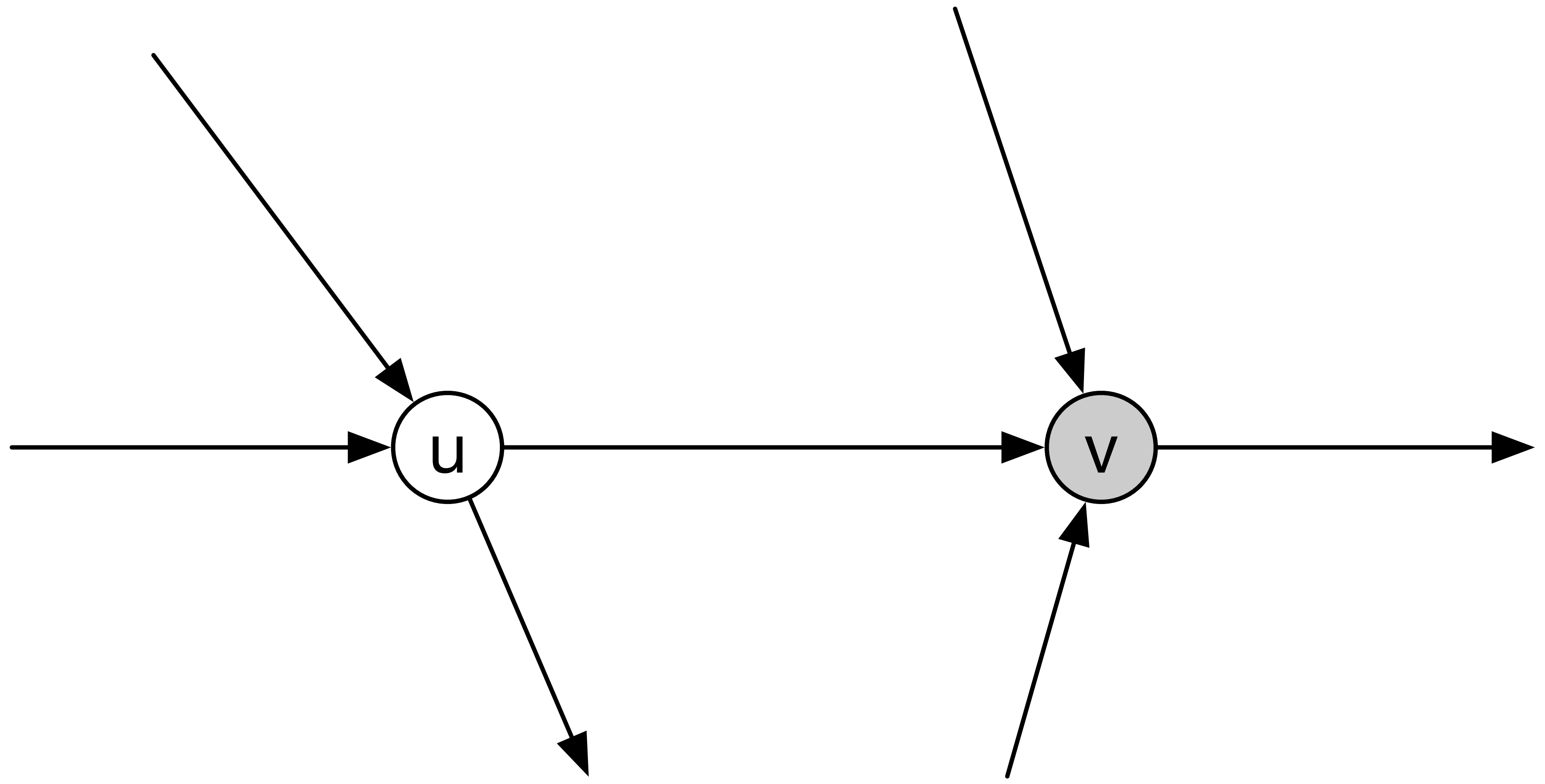


Semi Matching

- **Orient** the edges of a graph such that, for each edge (u, v) oriented from u to v, it holds that $\deg_{\text{in}}(v) \leq \deg_{\text{in}}(u) + 1$



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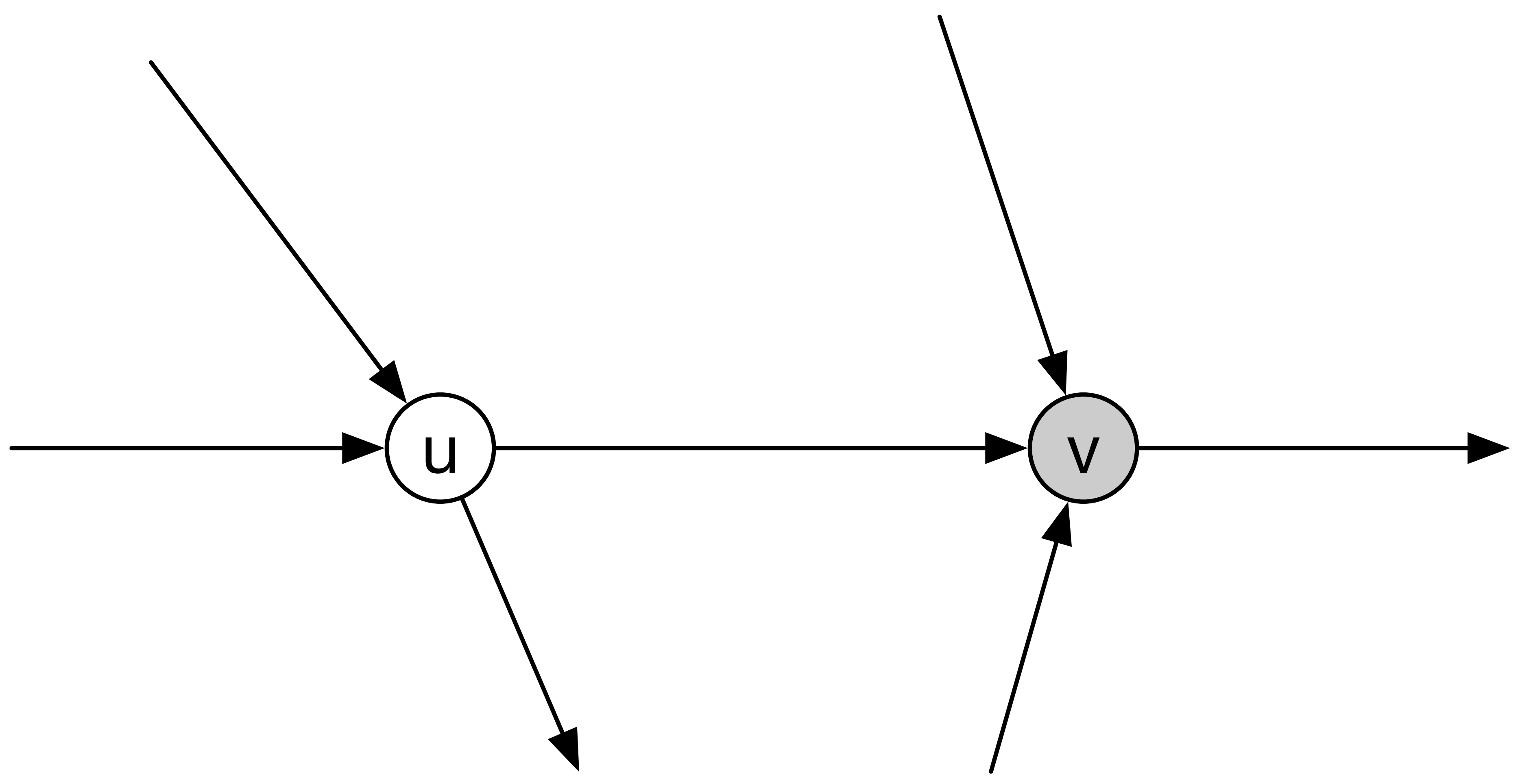

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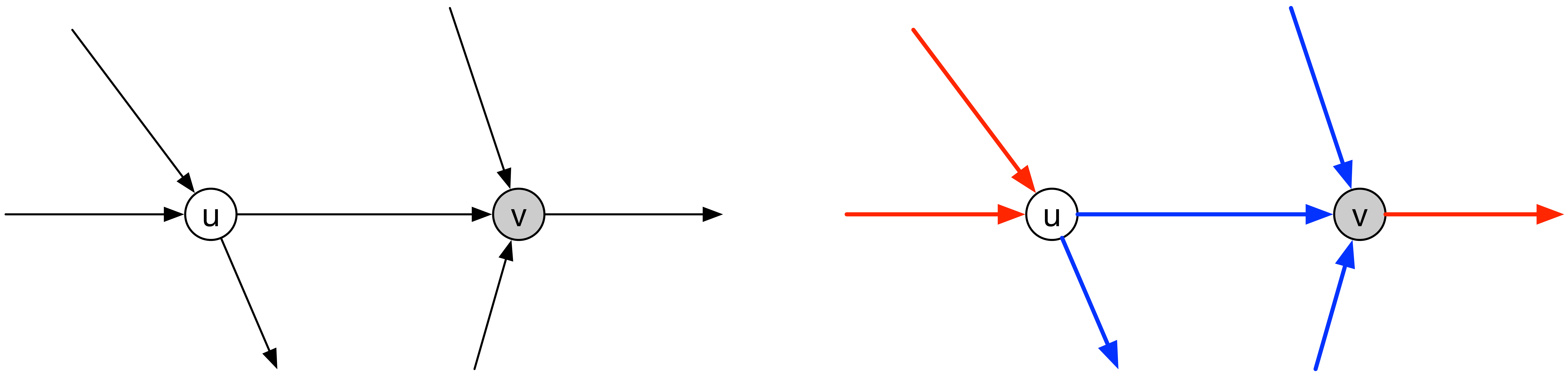

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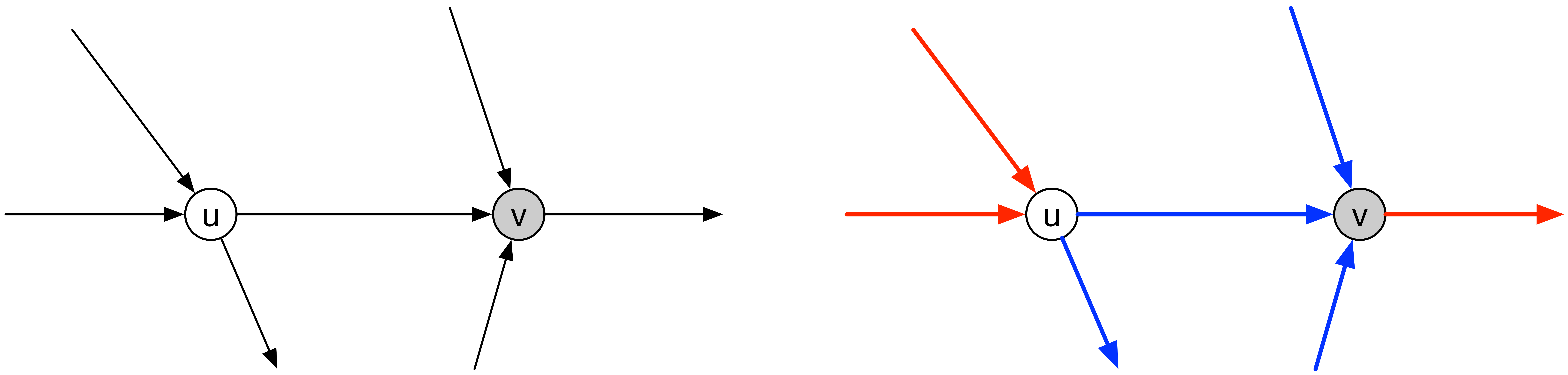

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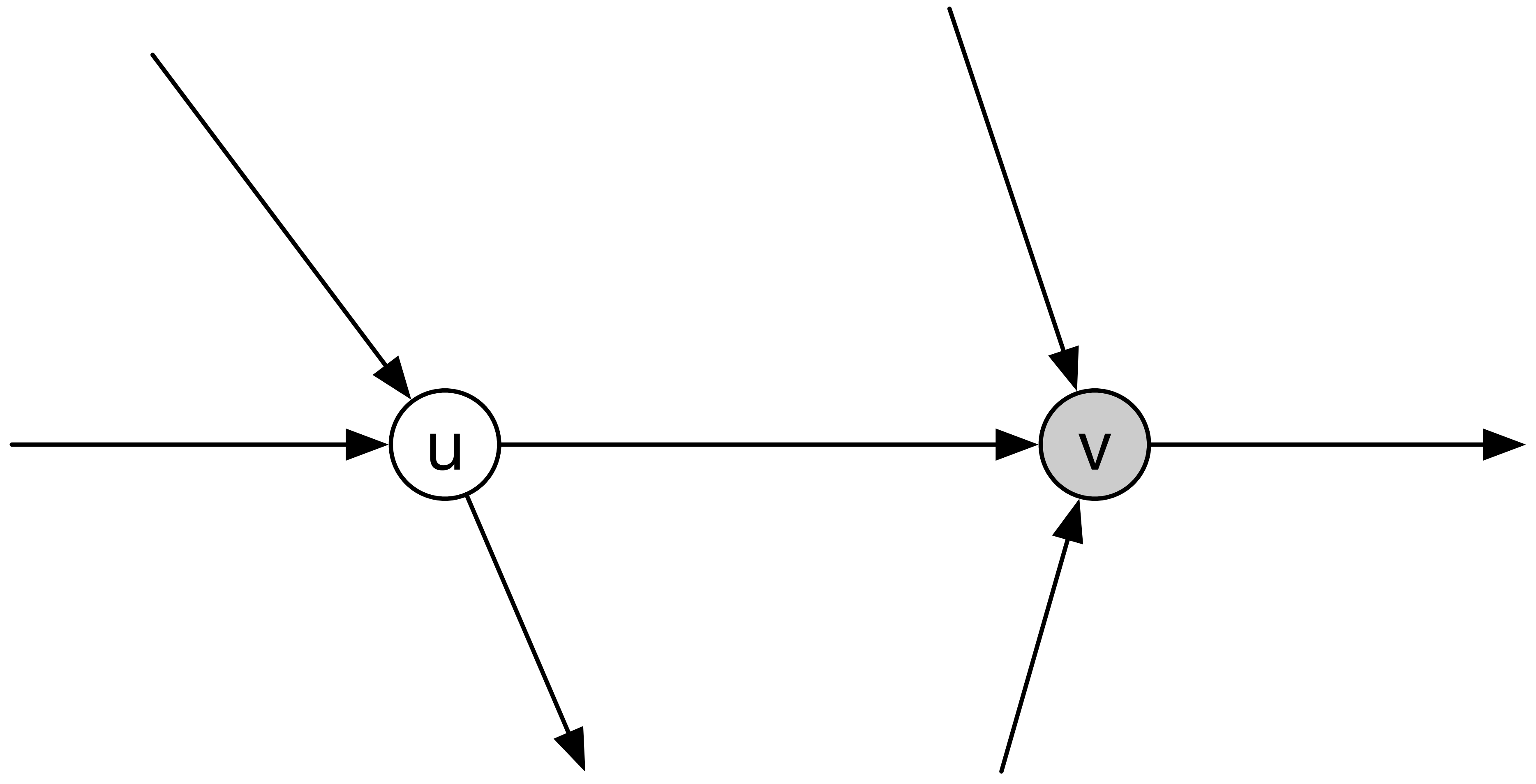

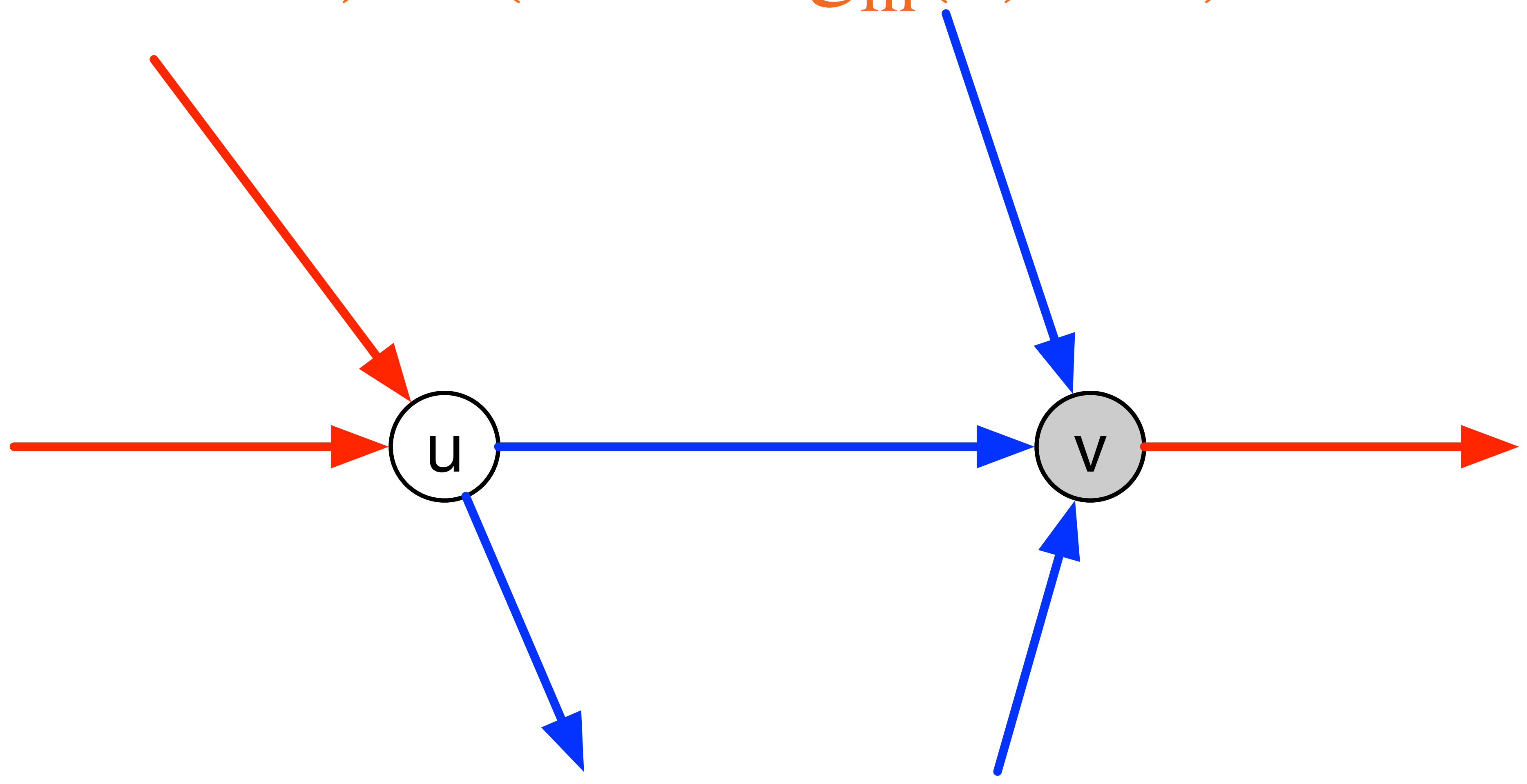

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Efficient Load-Balancing through Distributed Token Dropping

[Brandt, Keller, Rybicki, Suomela, Uitto 2021]

This problem can be solved in $O(\Delta^4)$ rounds!

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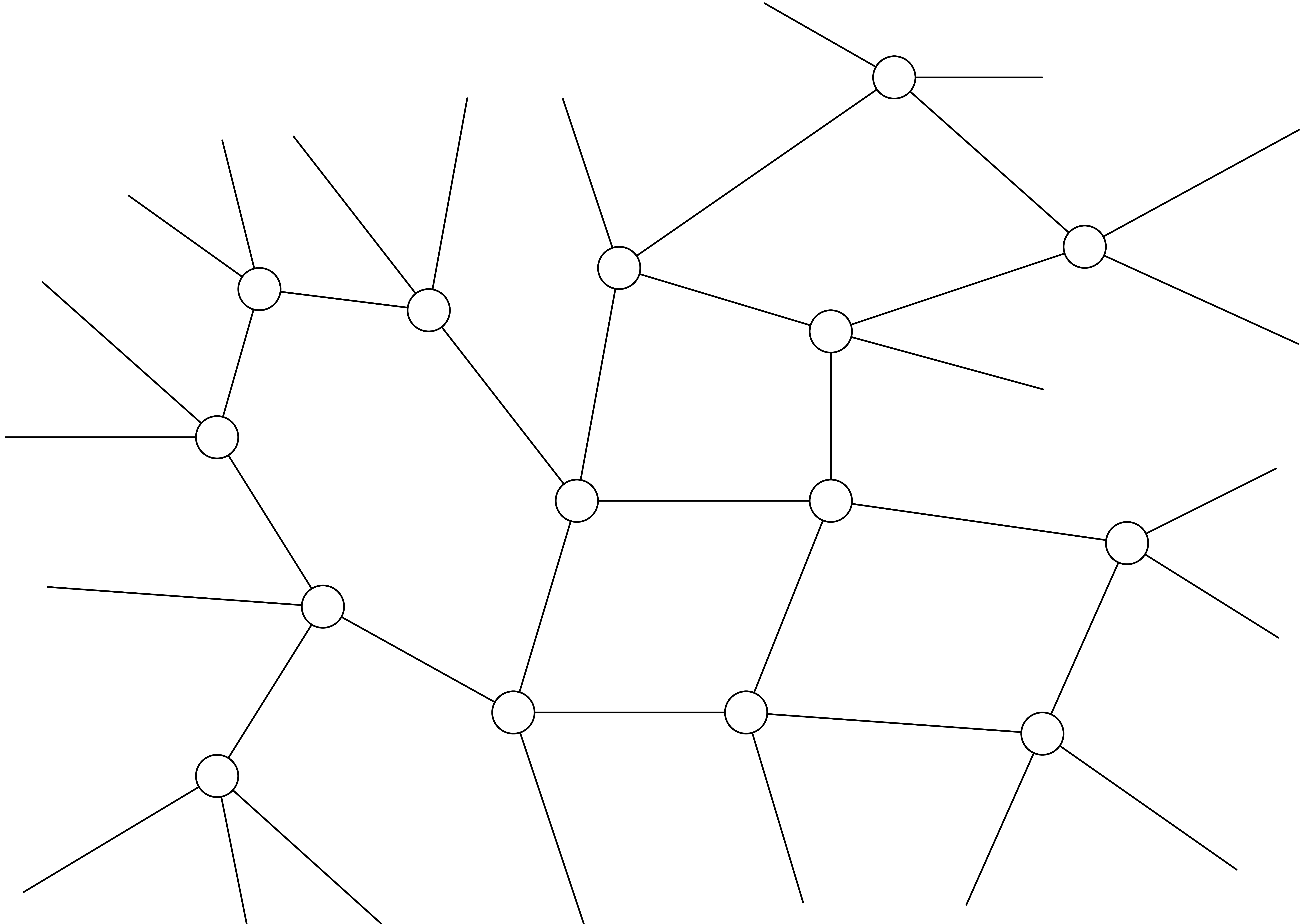

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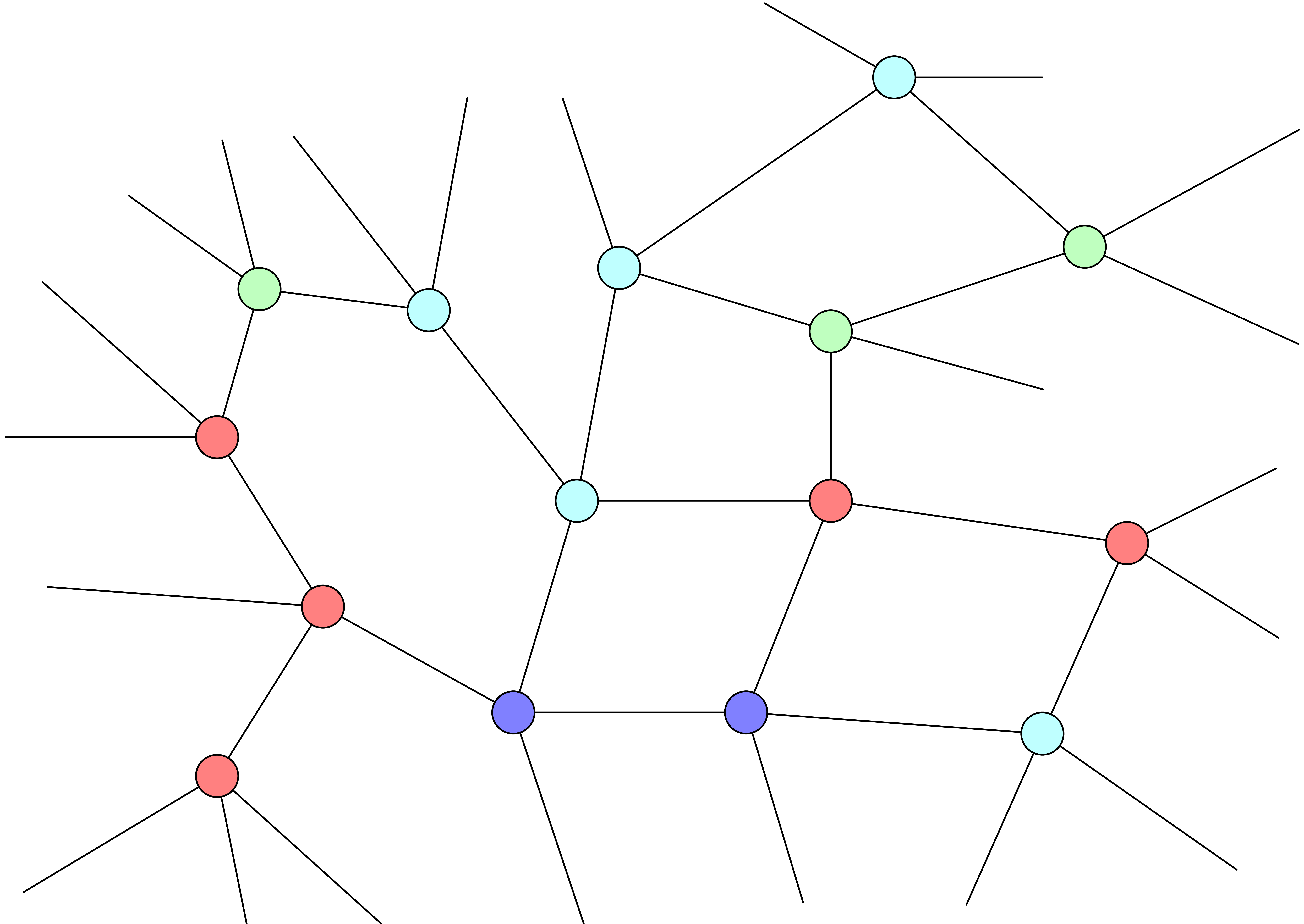

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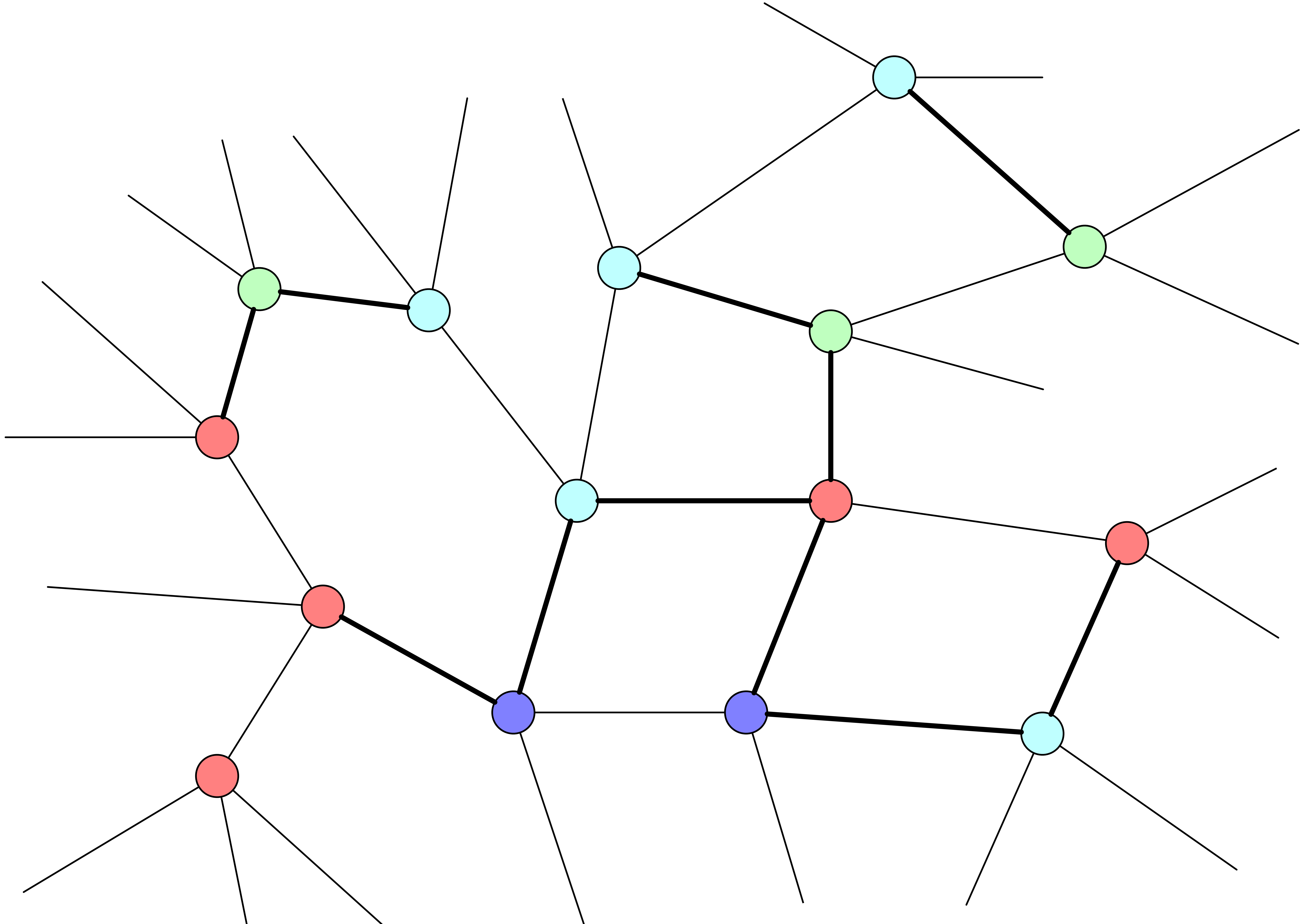

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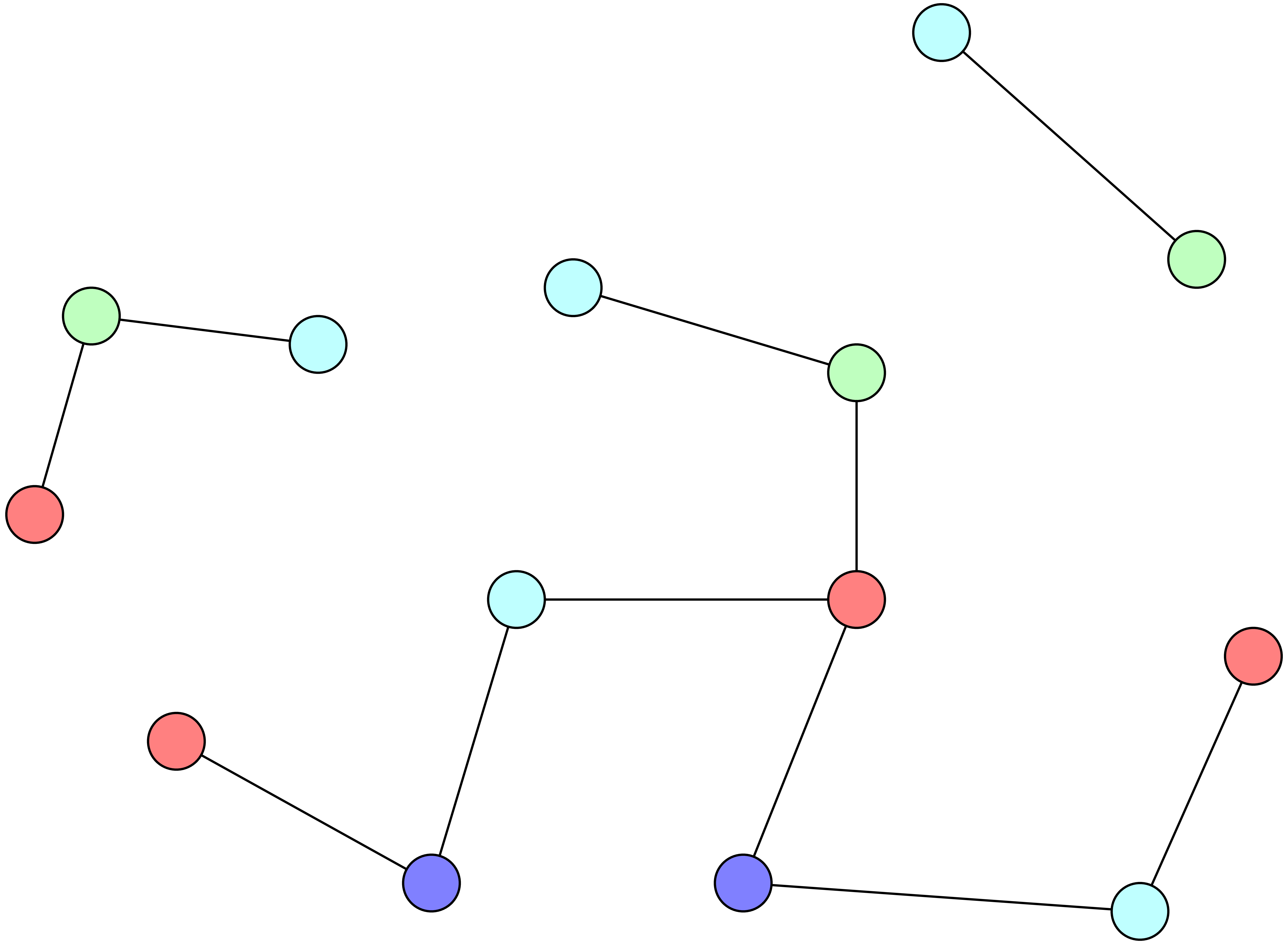

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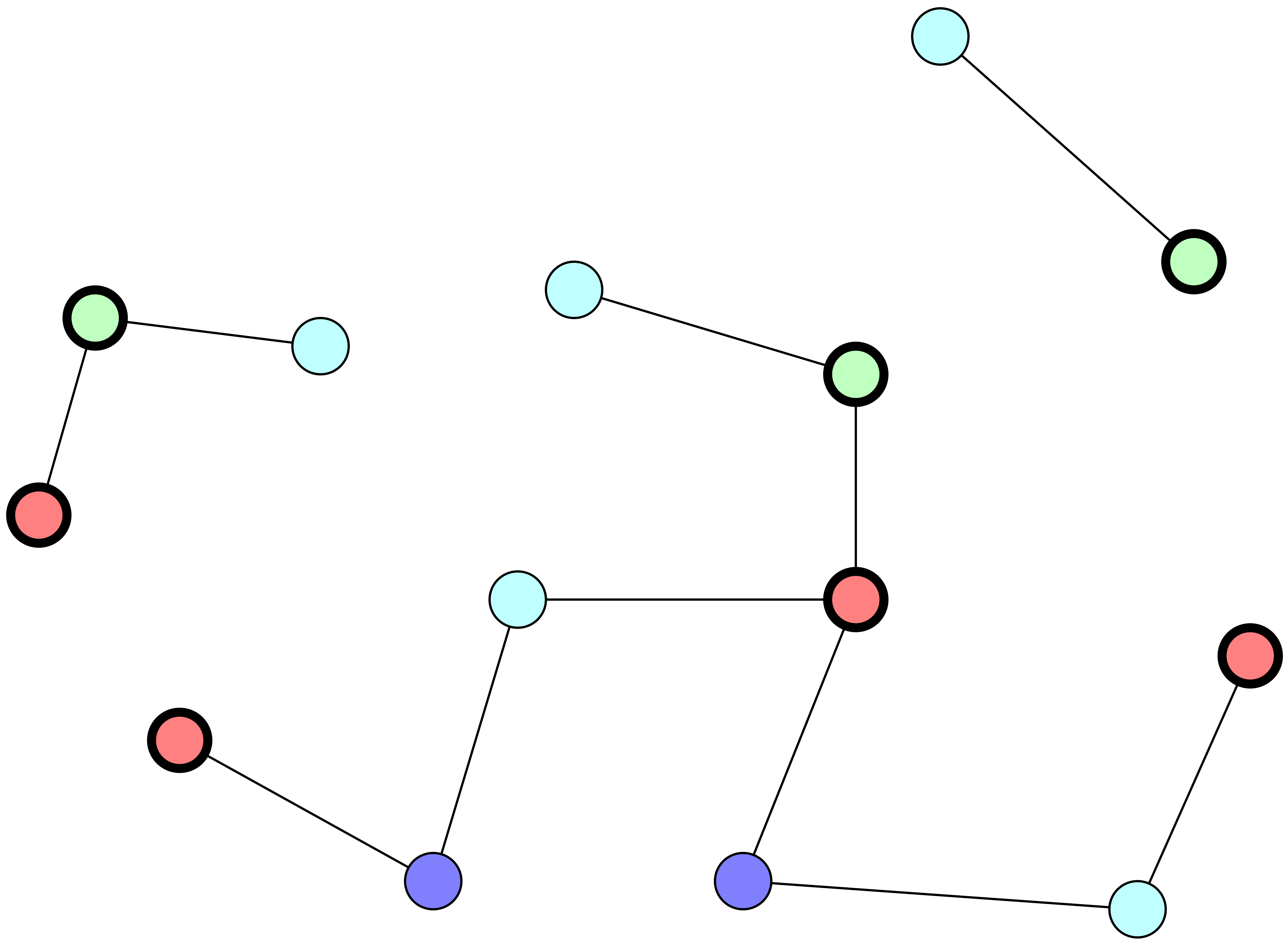

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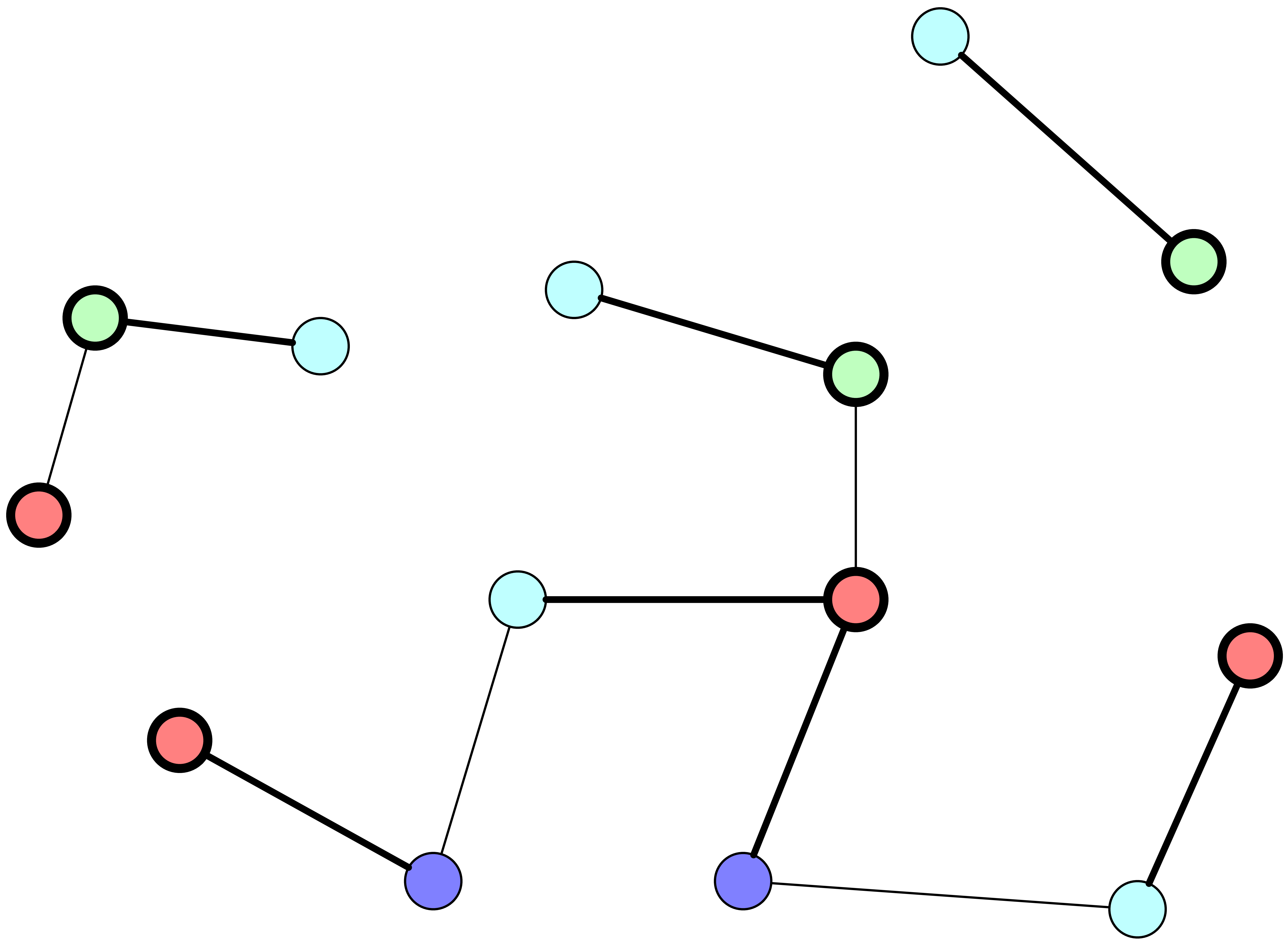

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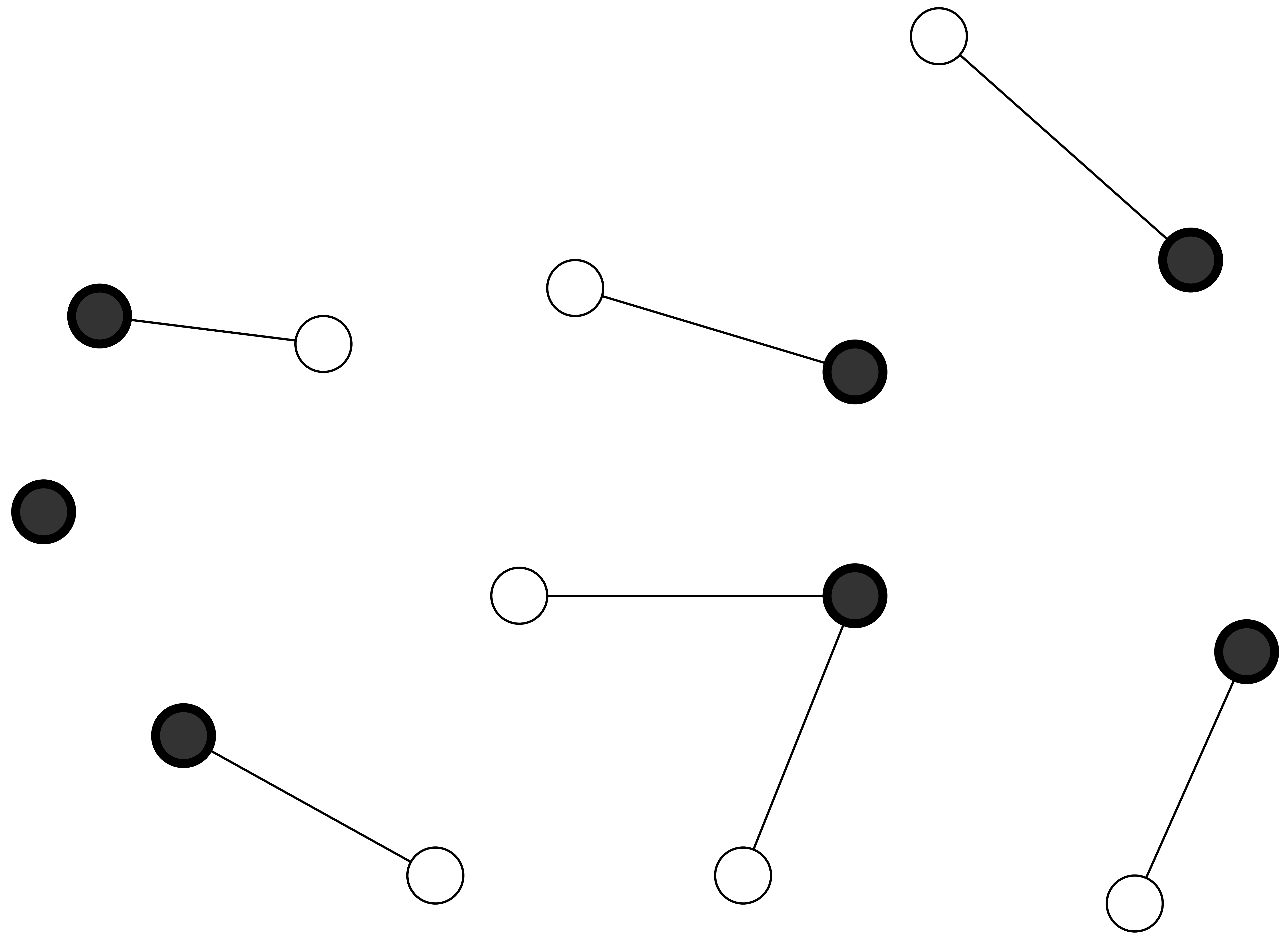

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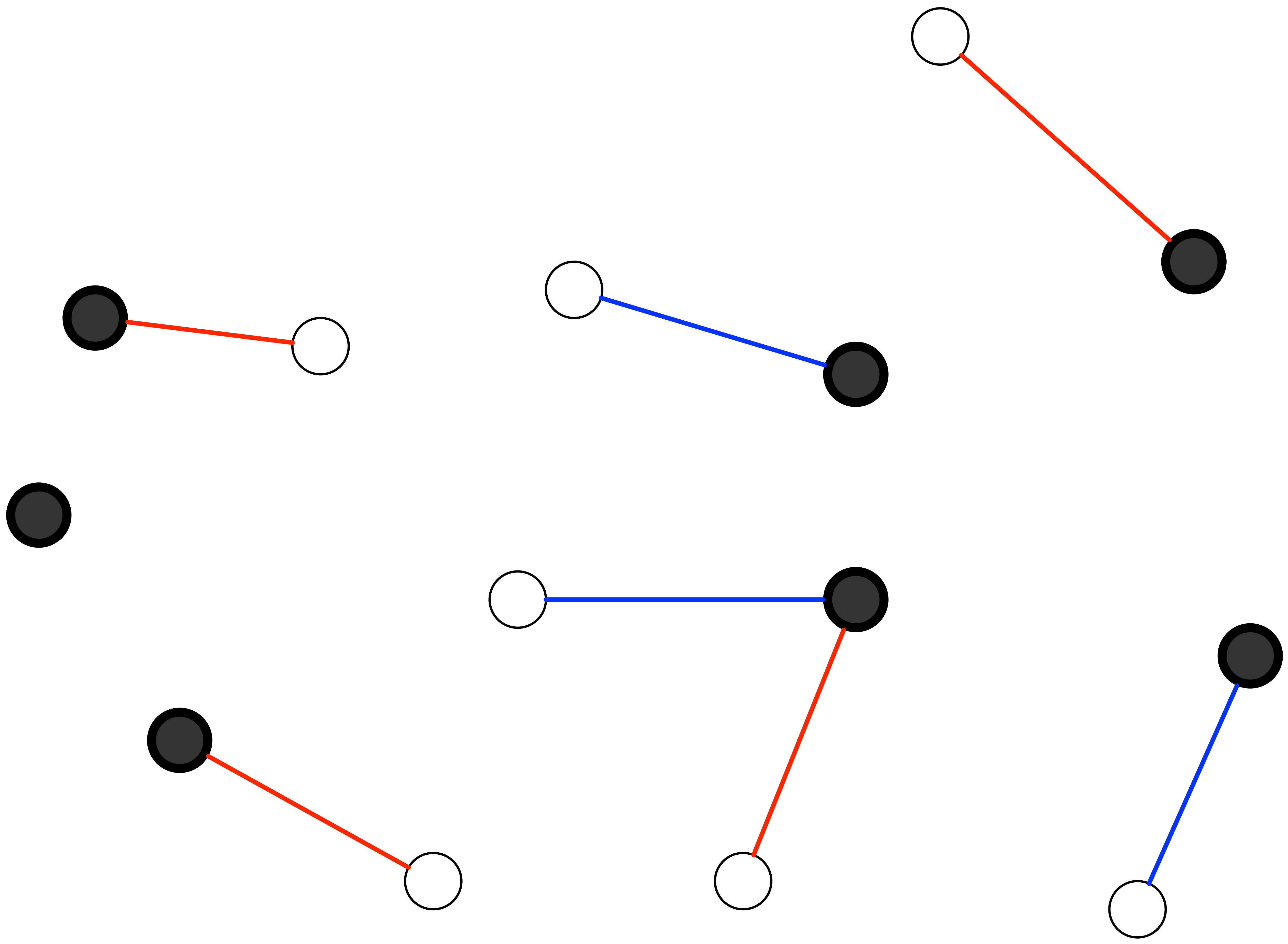

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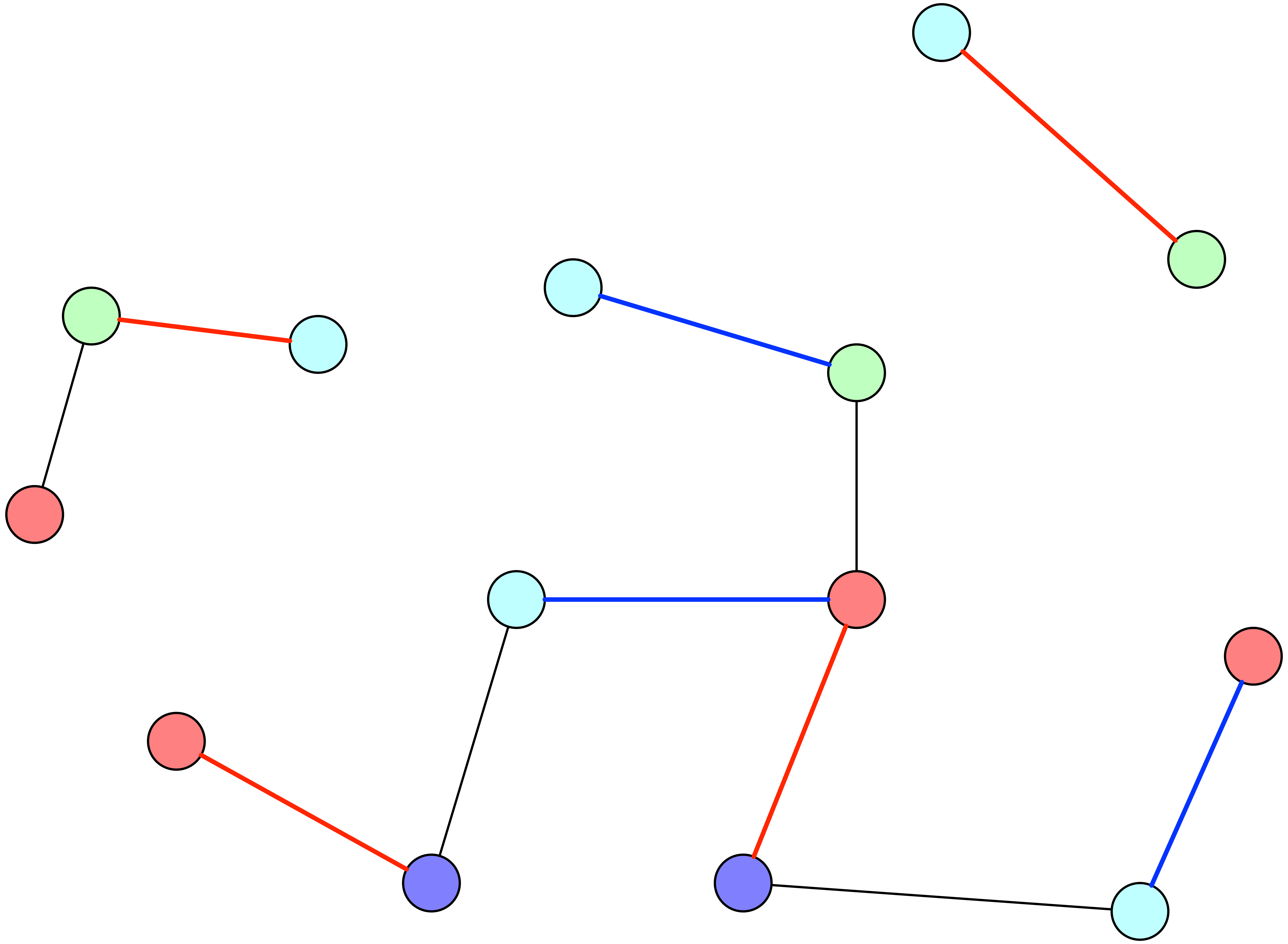

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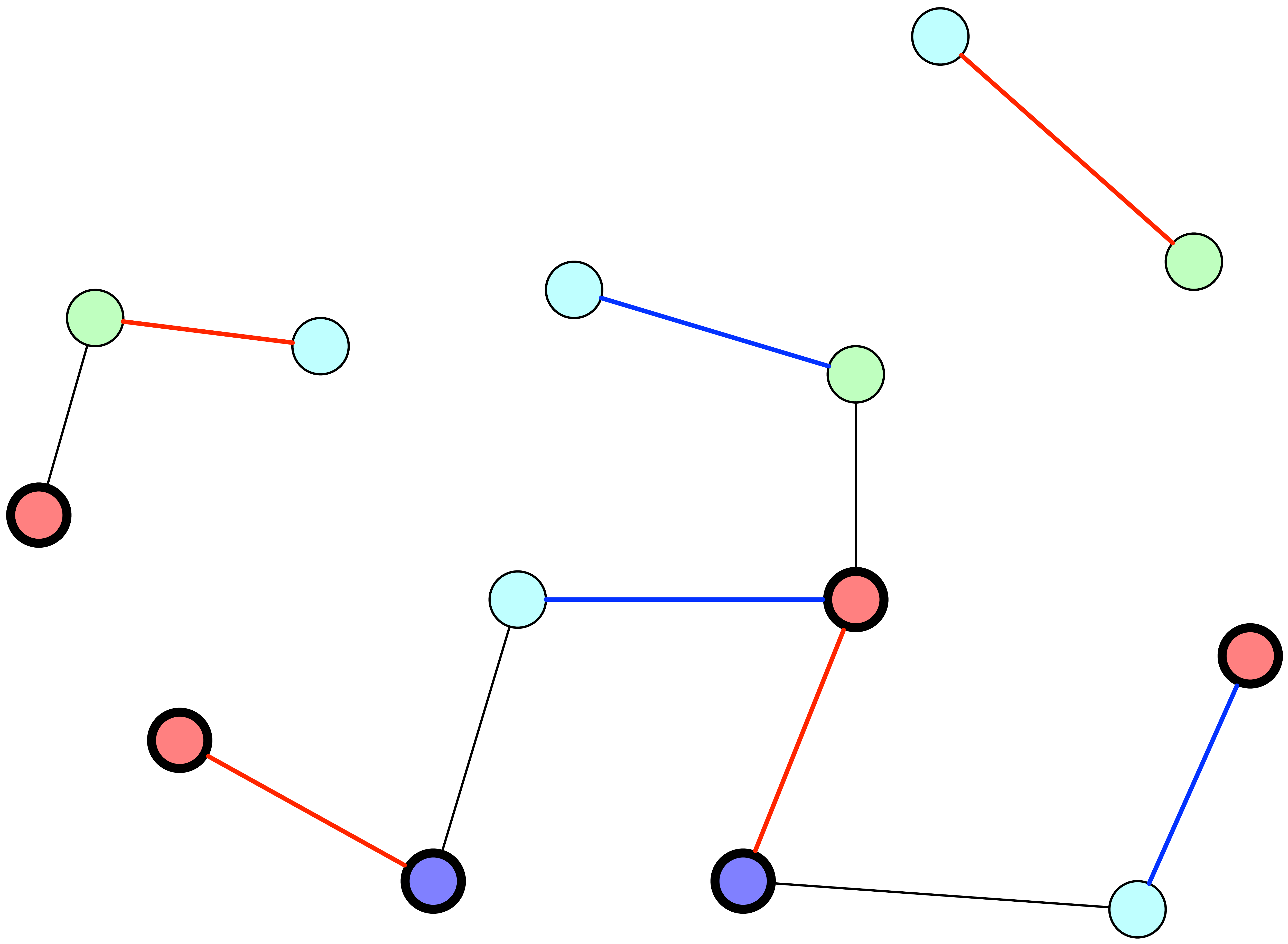

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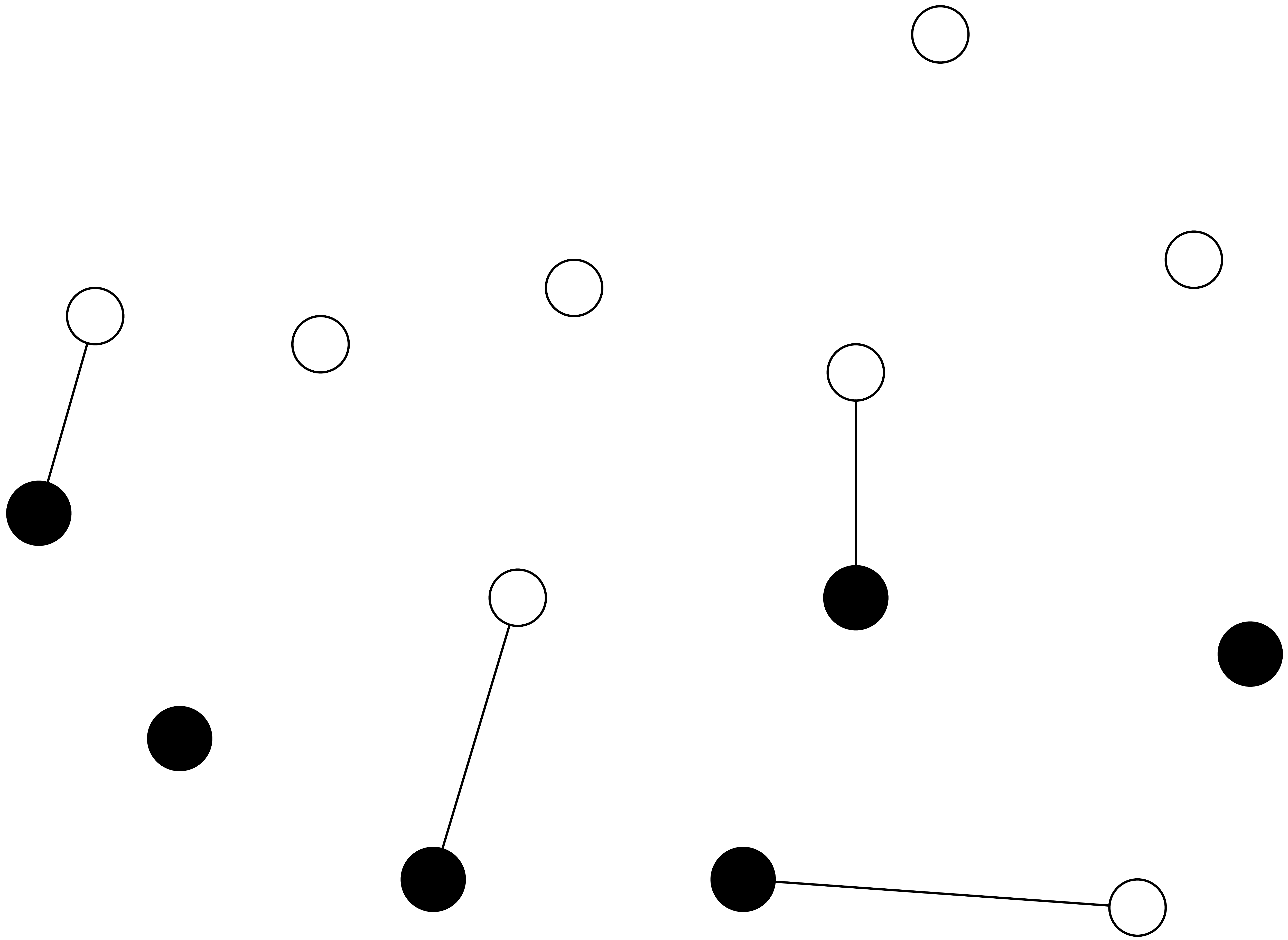

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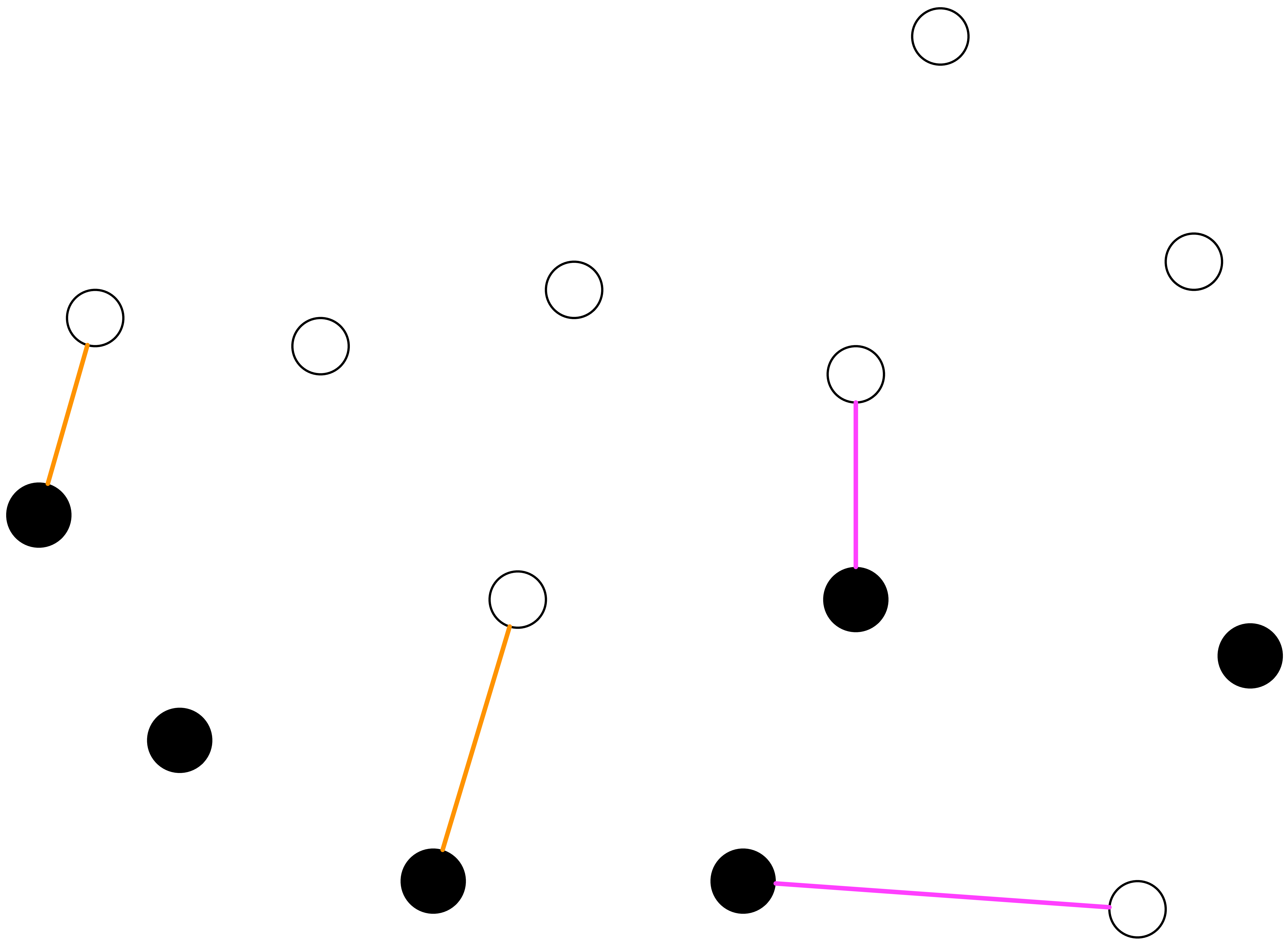

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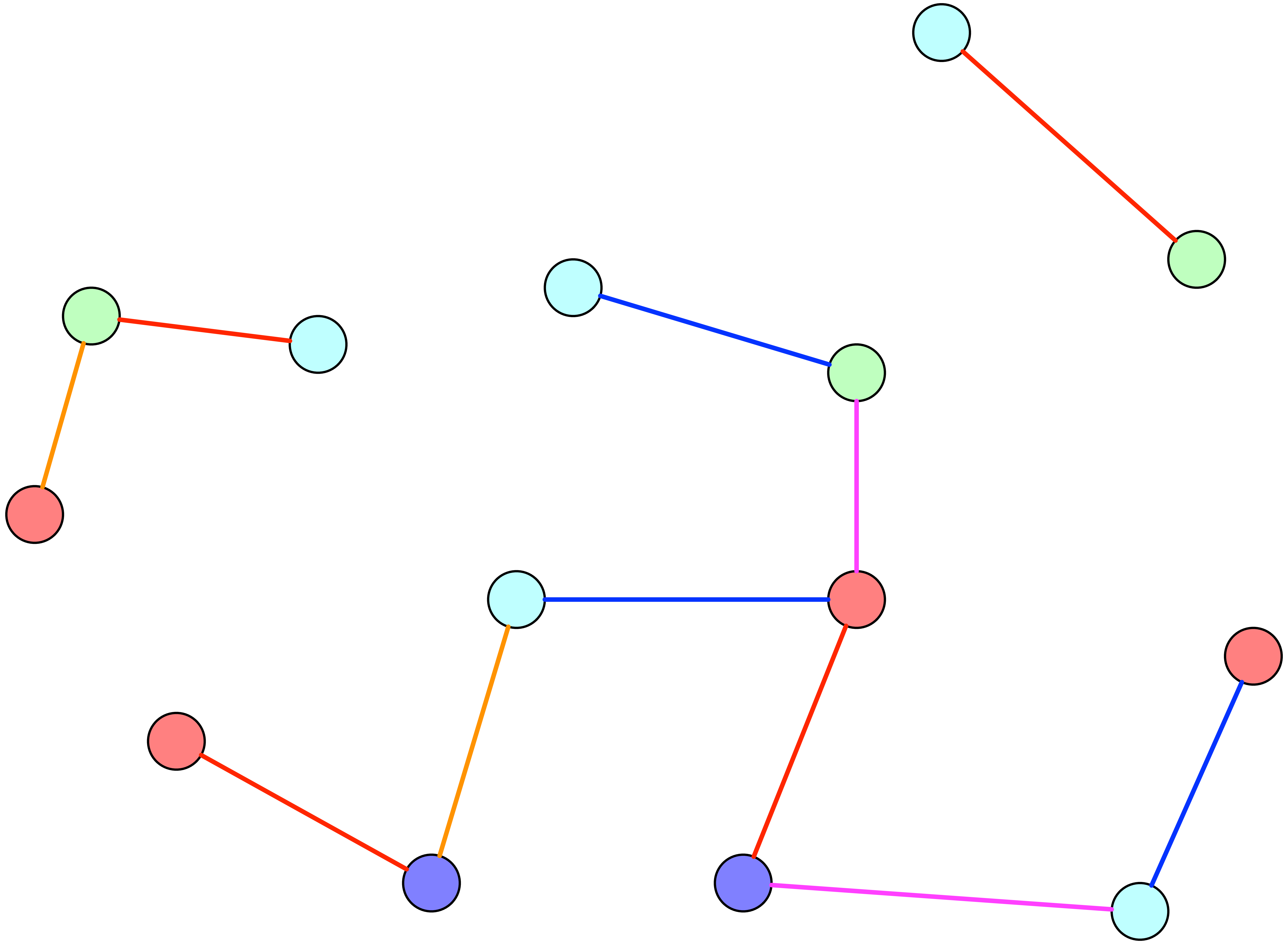

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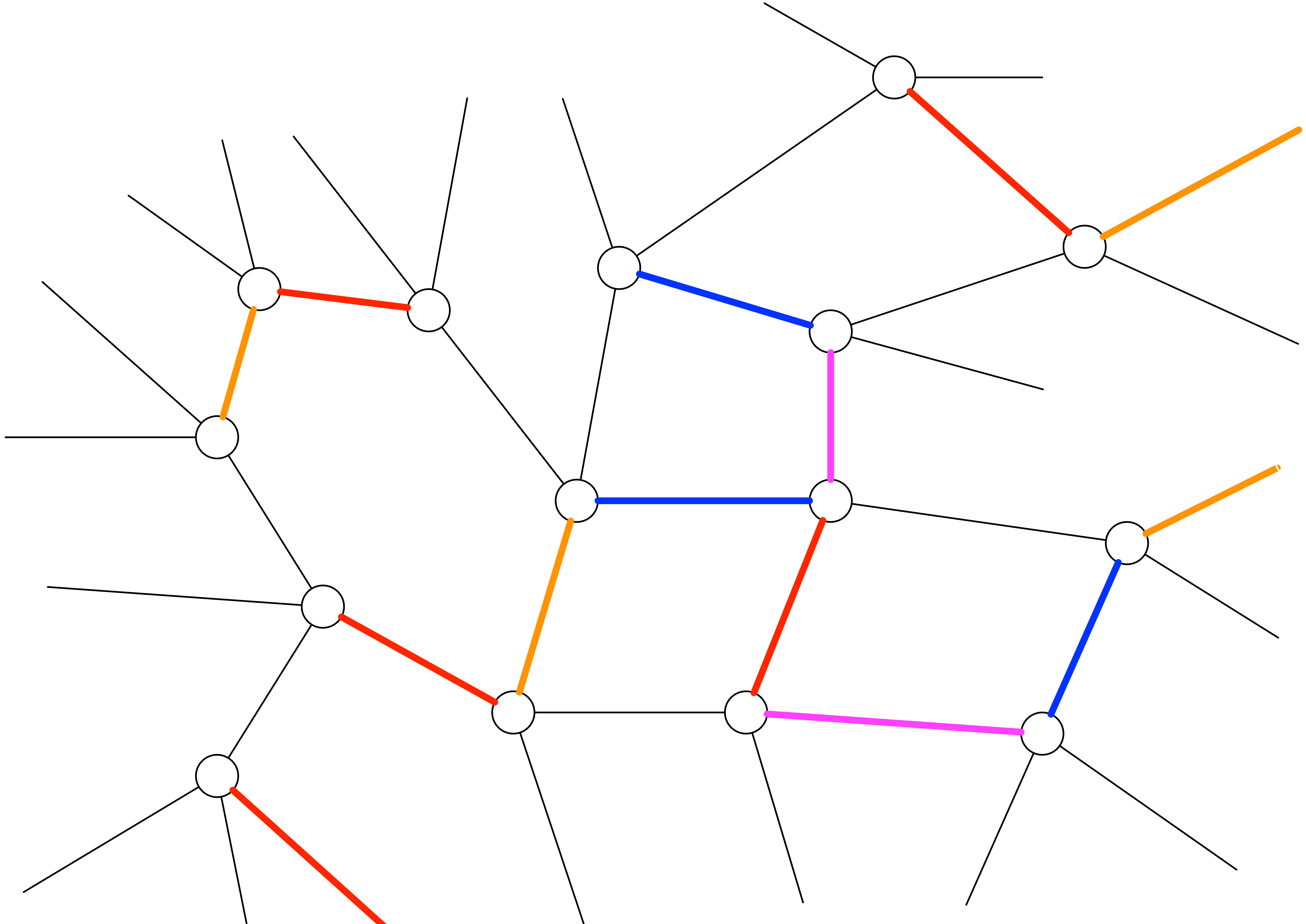

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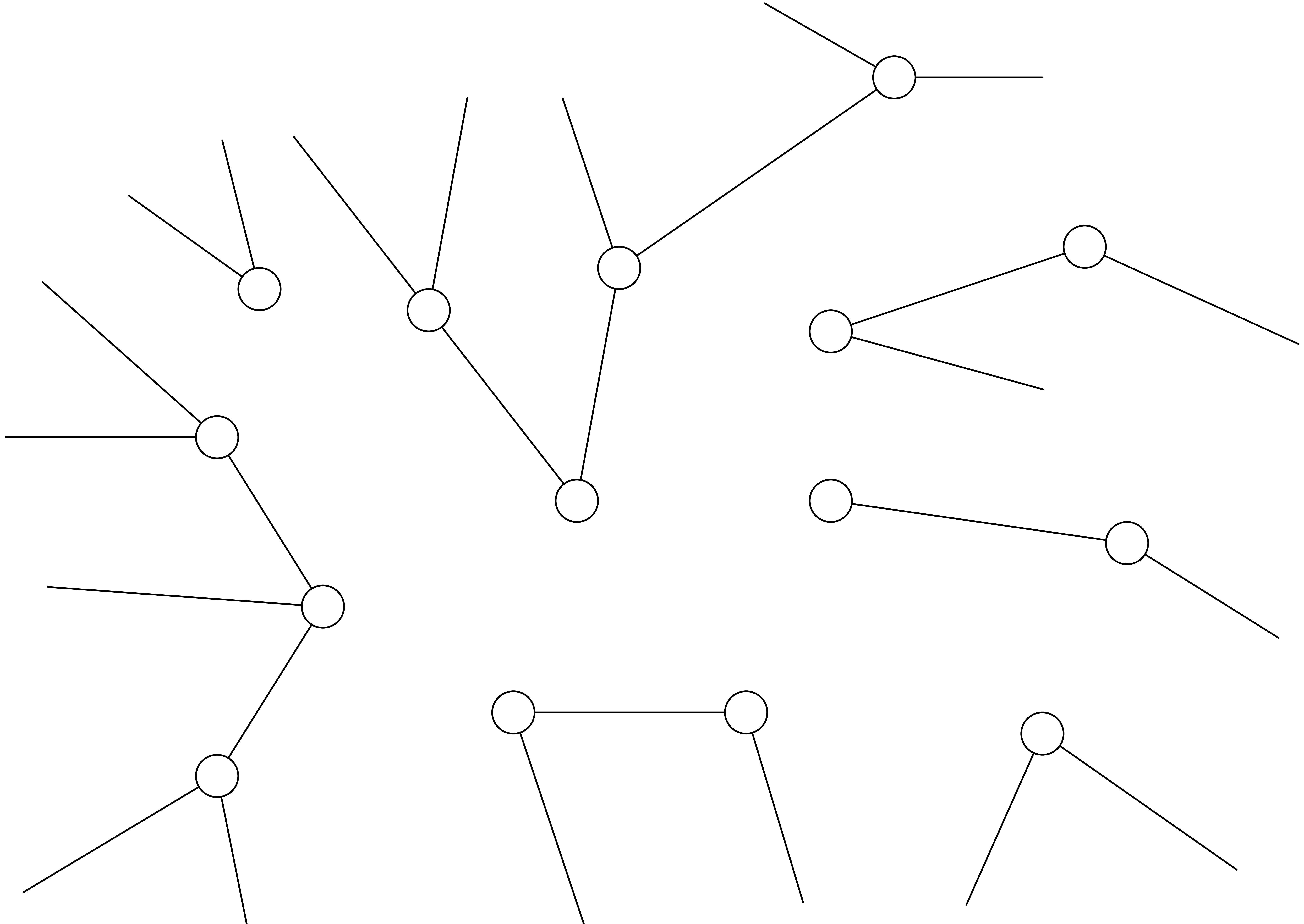

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Given a T round algorithm for $O(\Delta)$-edge coloring in bipartite 2-colored graphs, we can construct an algorithm for $O(\Delta)$-edge coloring in general graphs that runs in $O(T \log \Delta + \log^* n)$ rounds

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- In order to achieve this, it turns out that it is enough to solve a more relaxed variant of semi-matching, that satisfies $\deg_{\text{in}}(v) \leq \deg_{\text{in}}(u) + \dfrac{\Delta}{\log \Delta}$

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- For $k = \dfrac{\Delta}{\log \Delta}$, this gives an $O(\log^5 \Delta)$ round algorithm!

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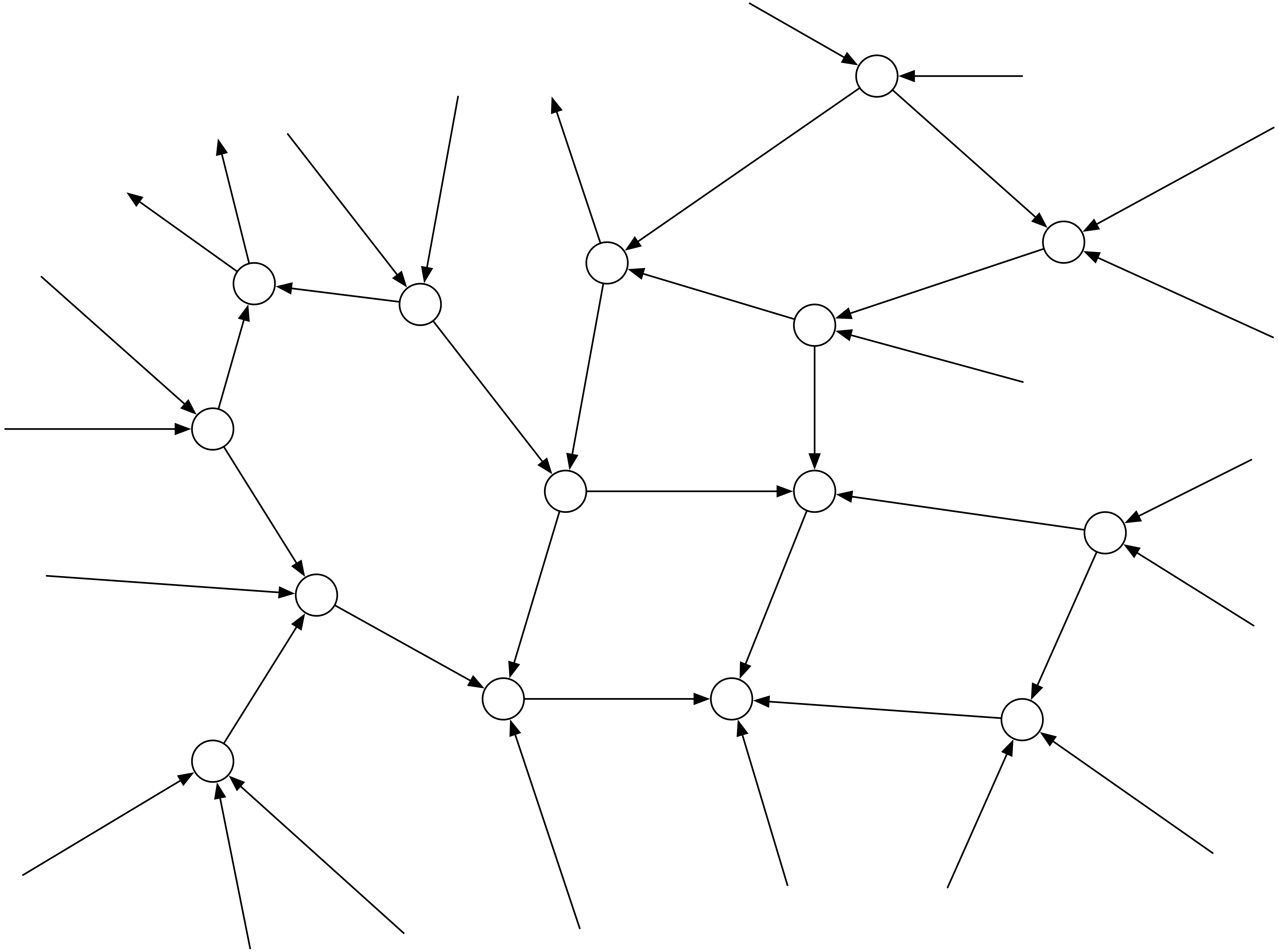

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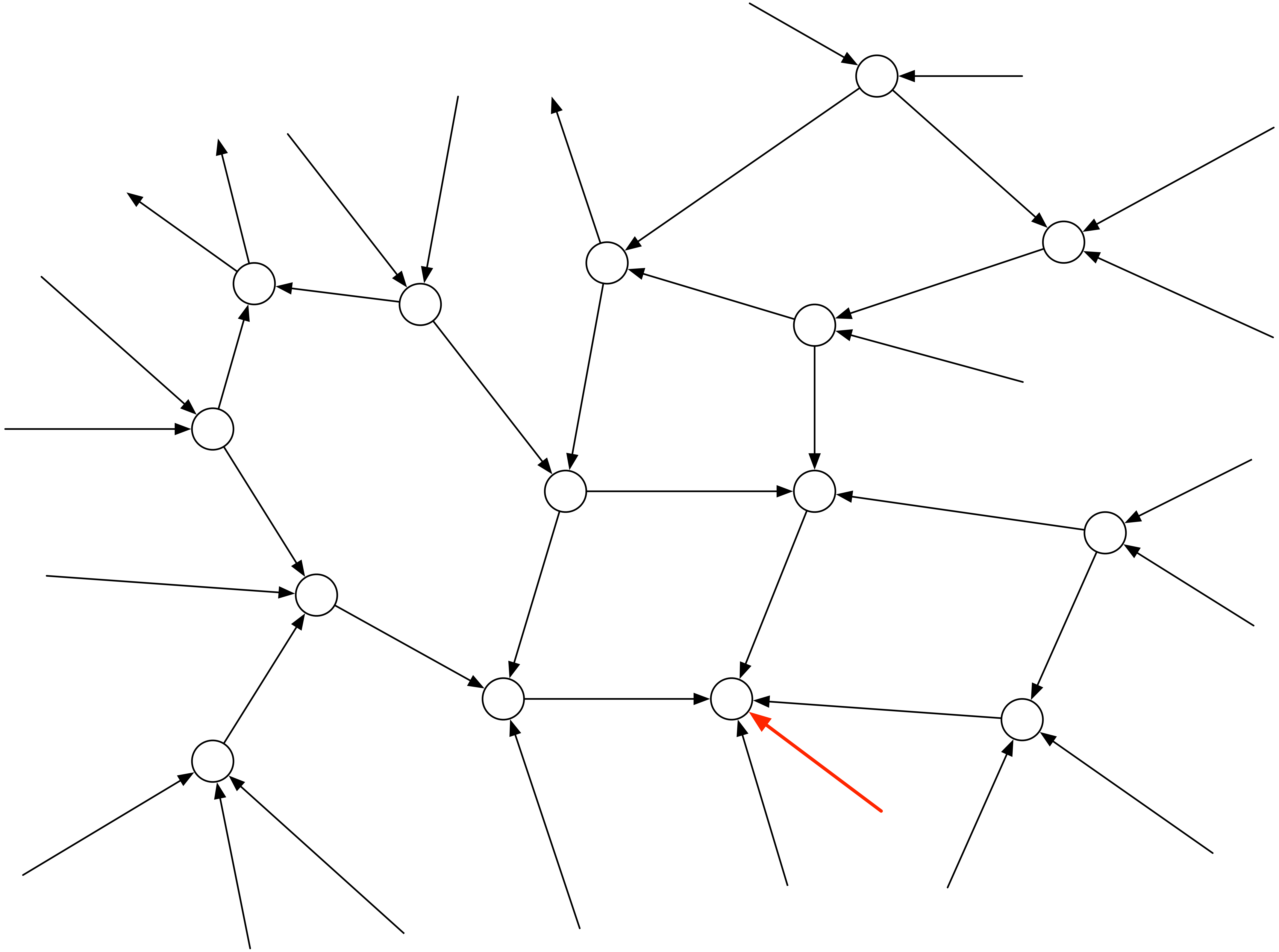

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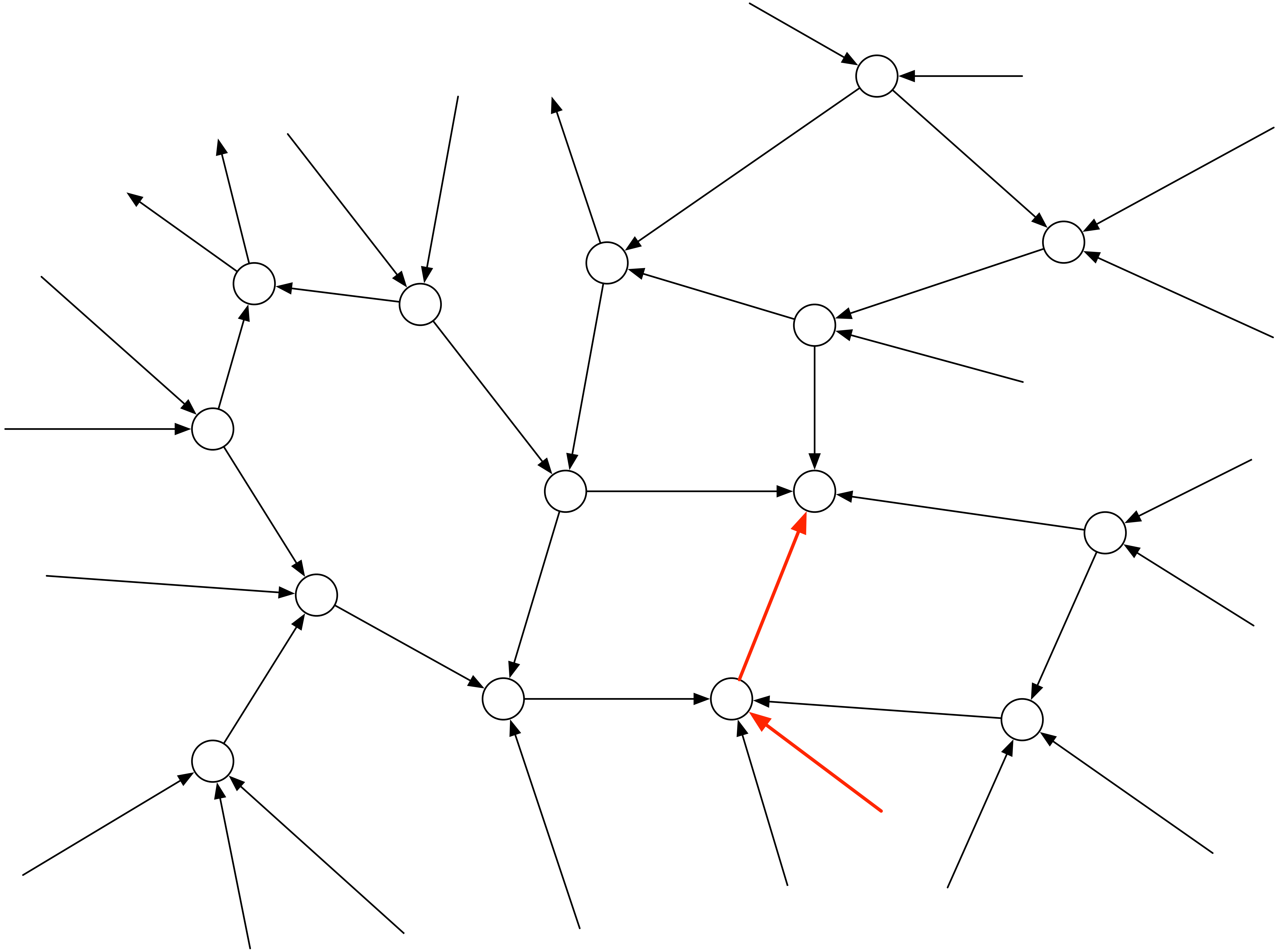

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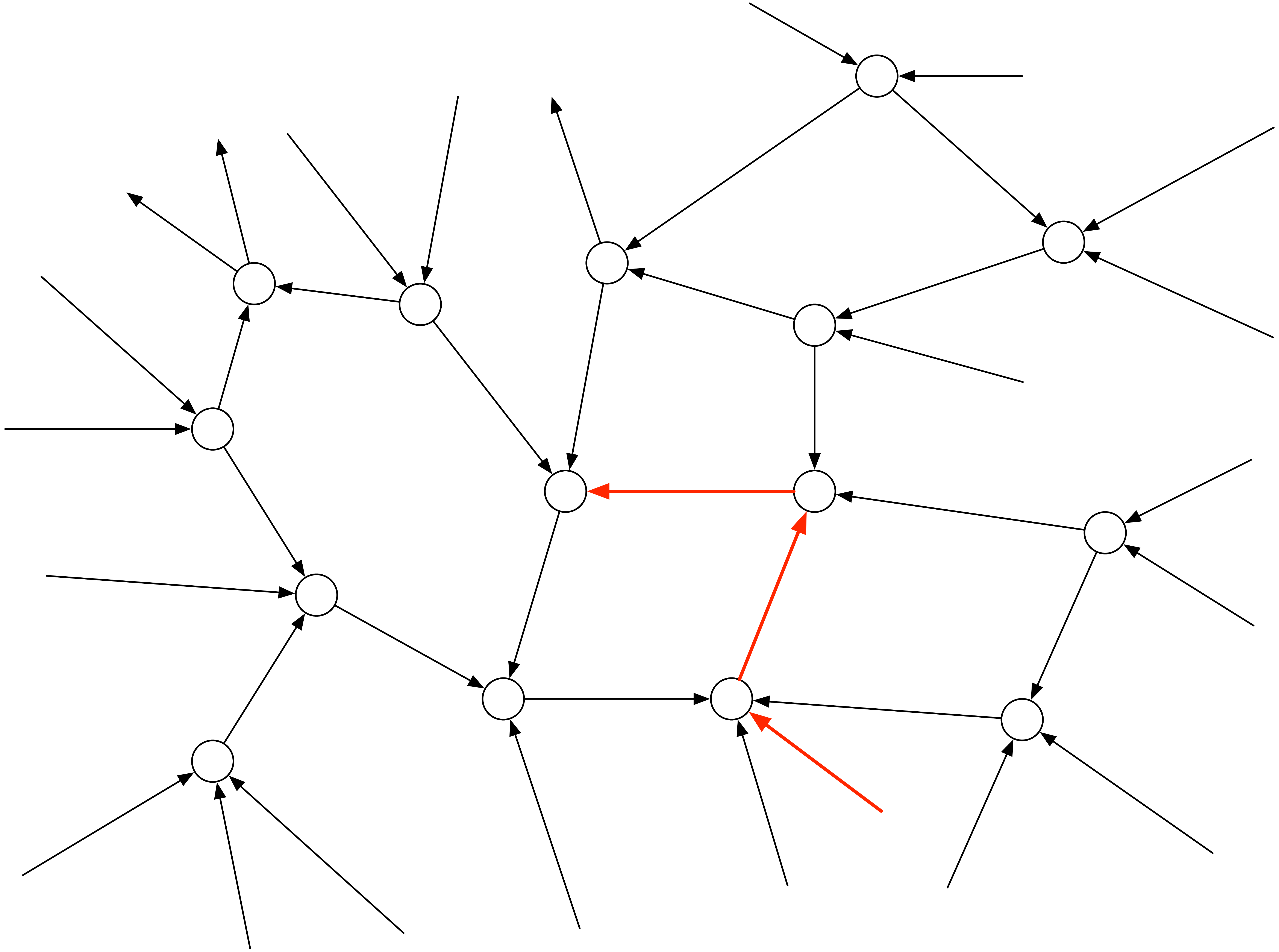

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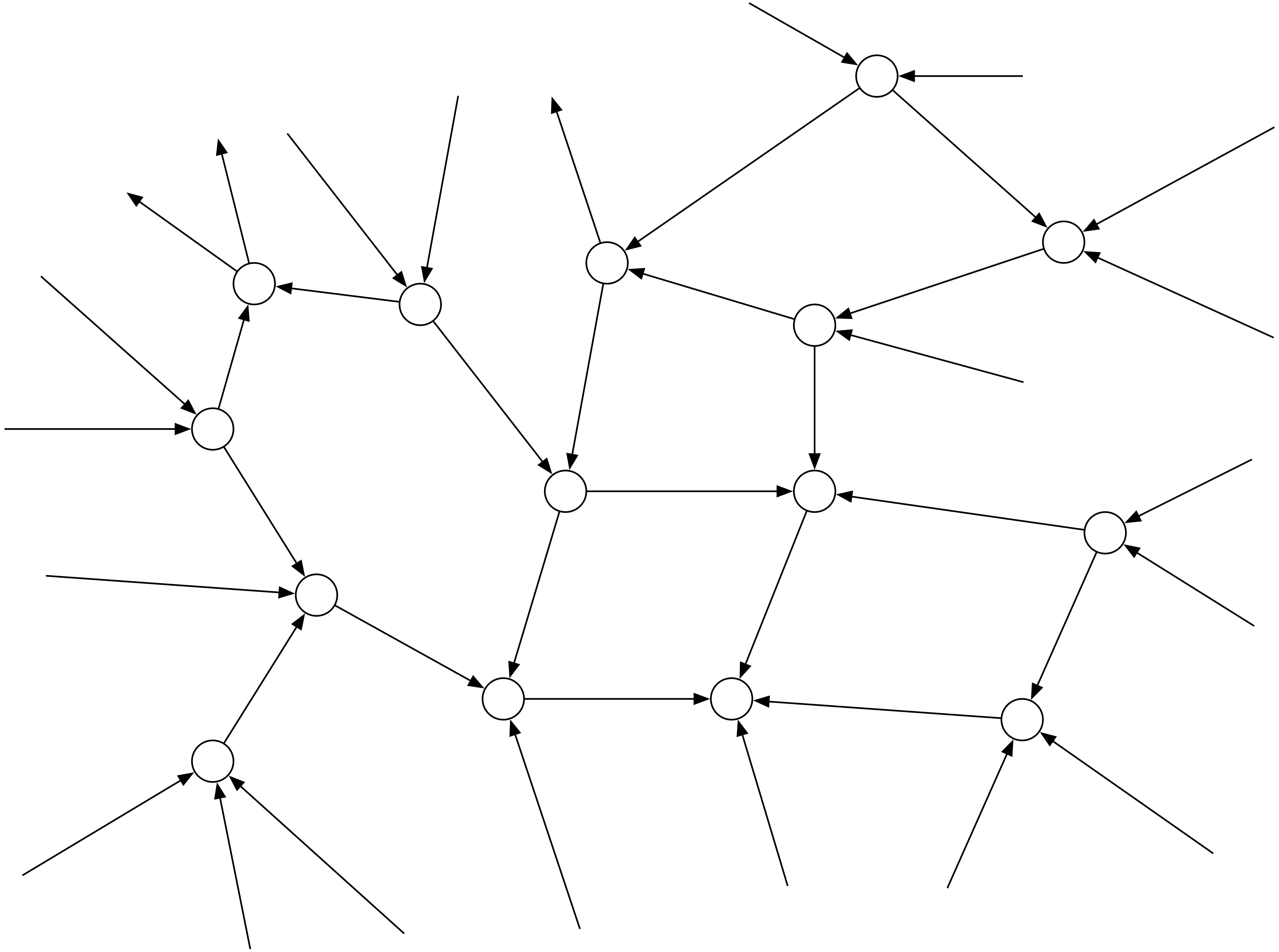

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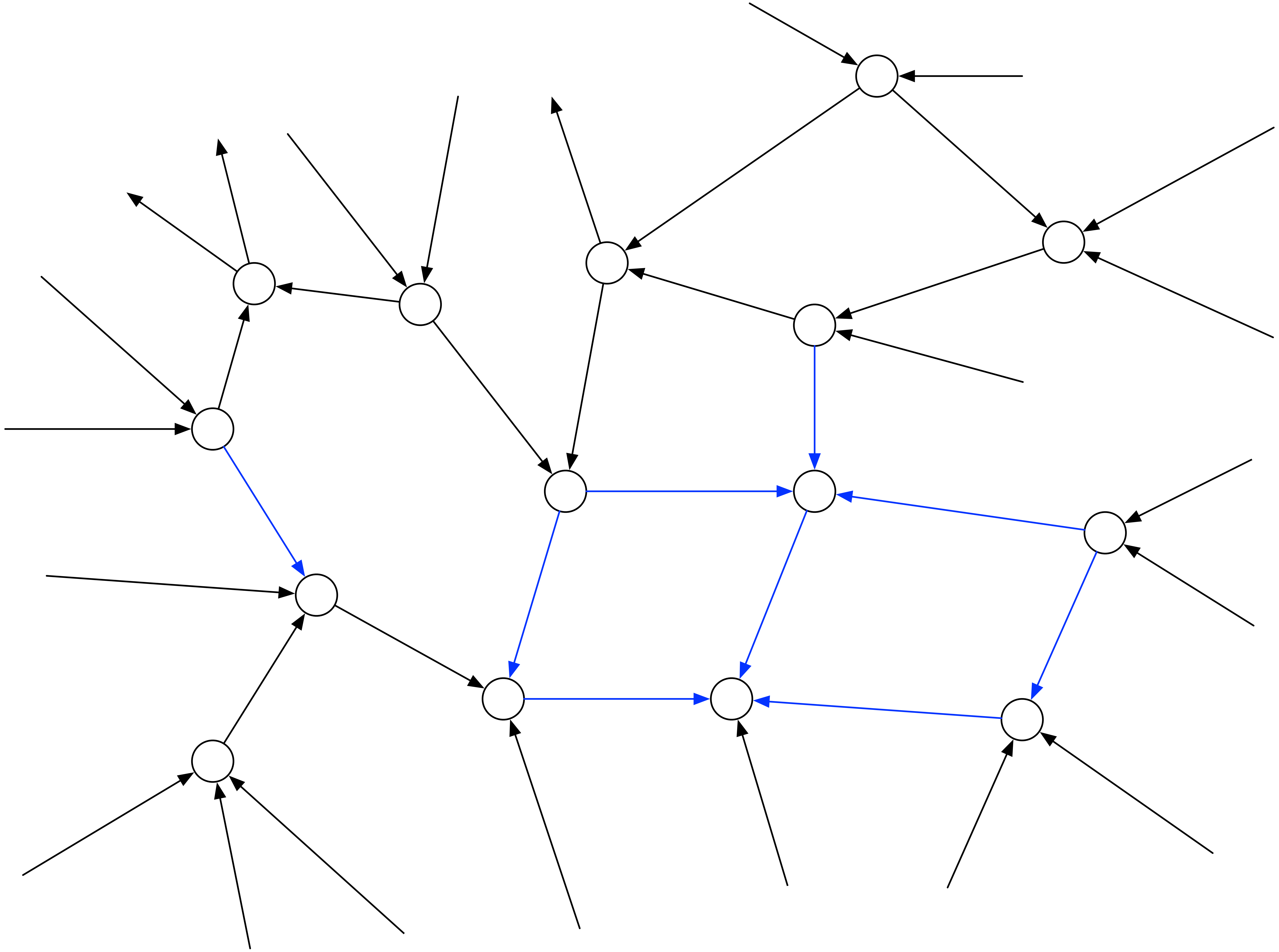

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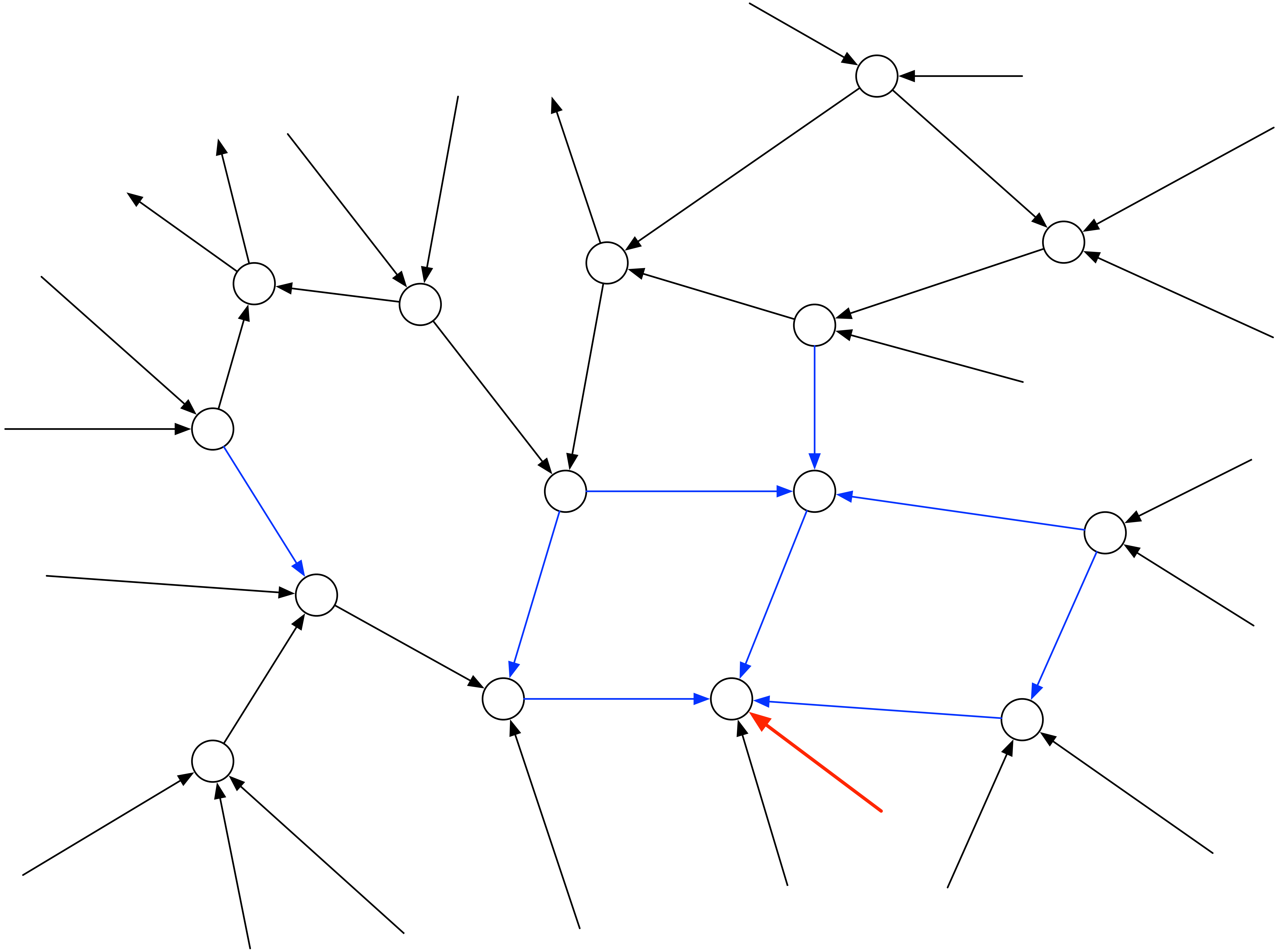

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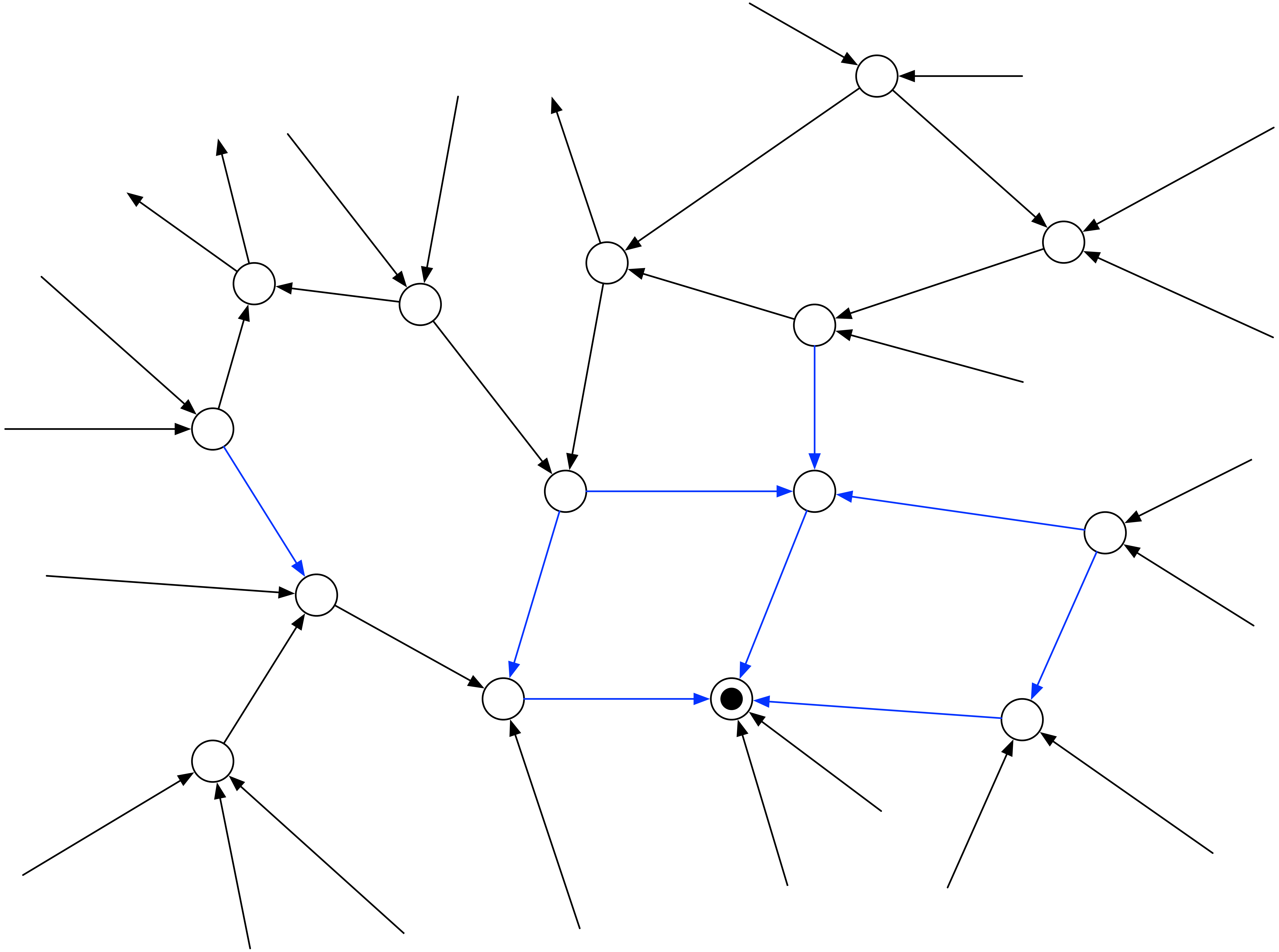

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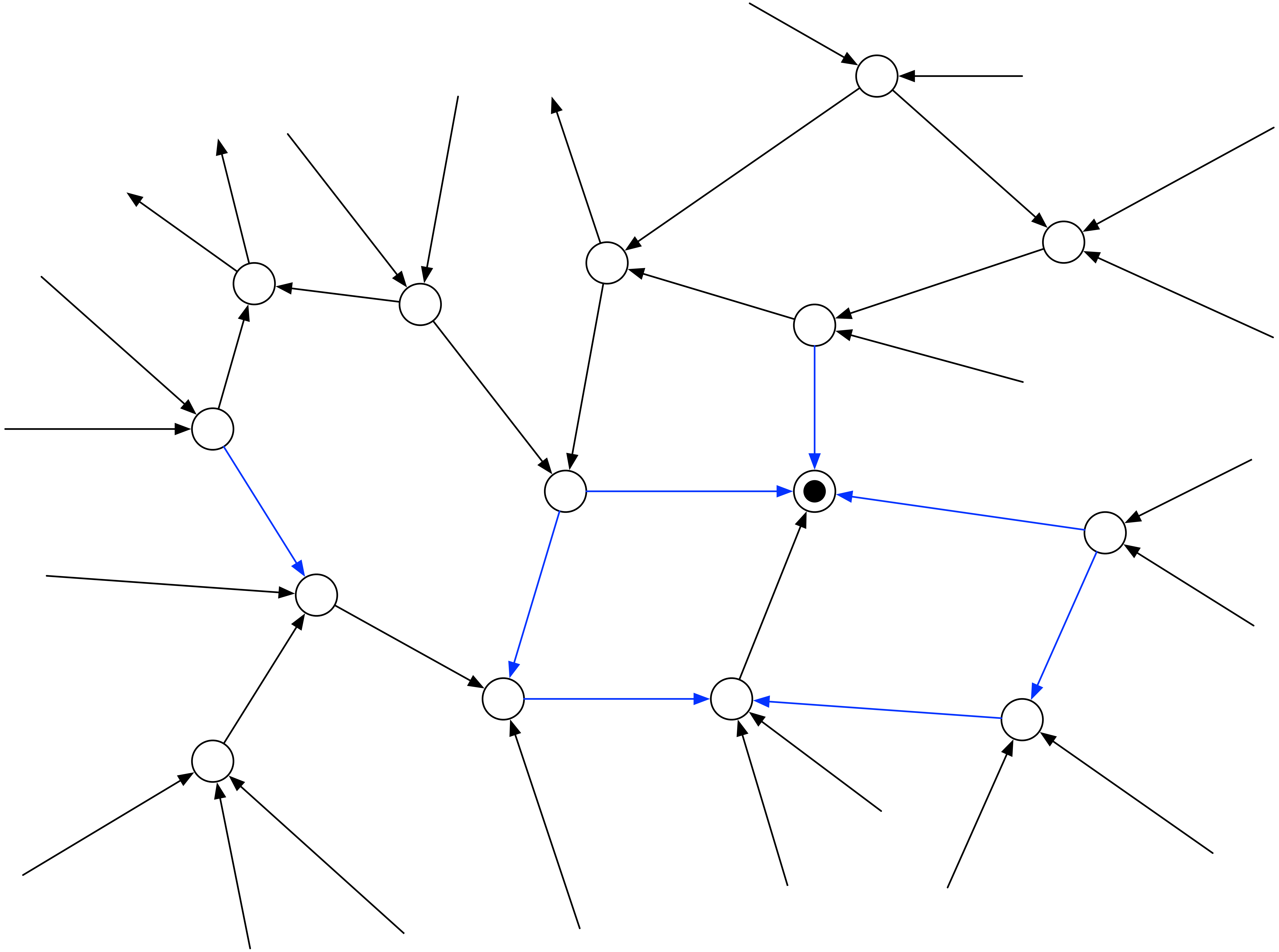

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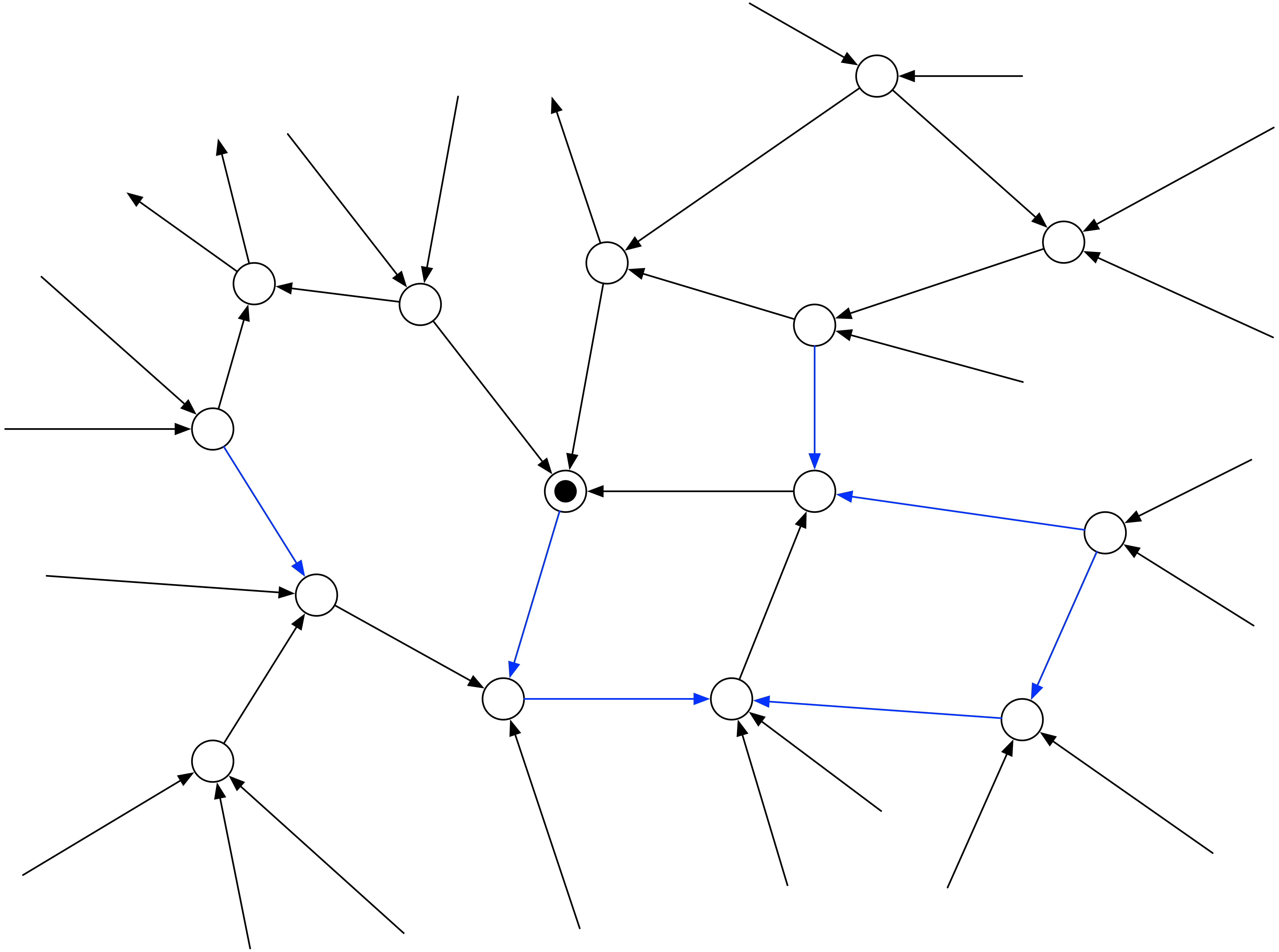
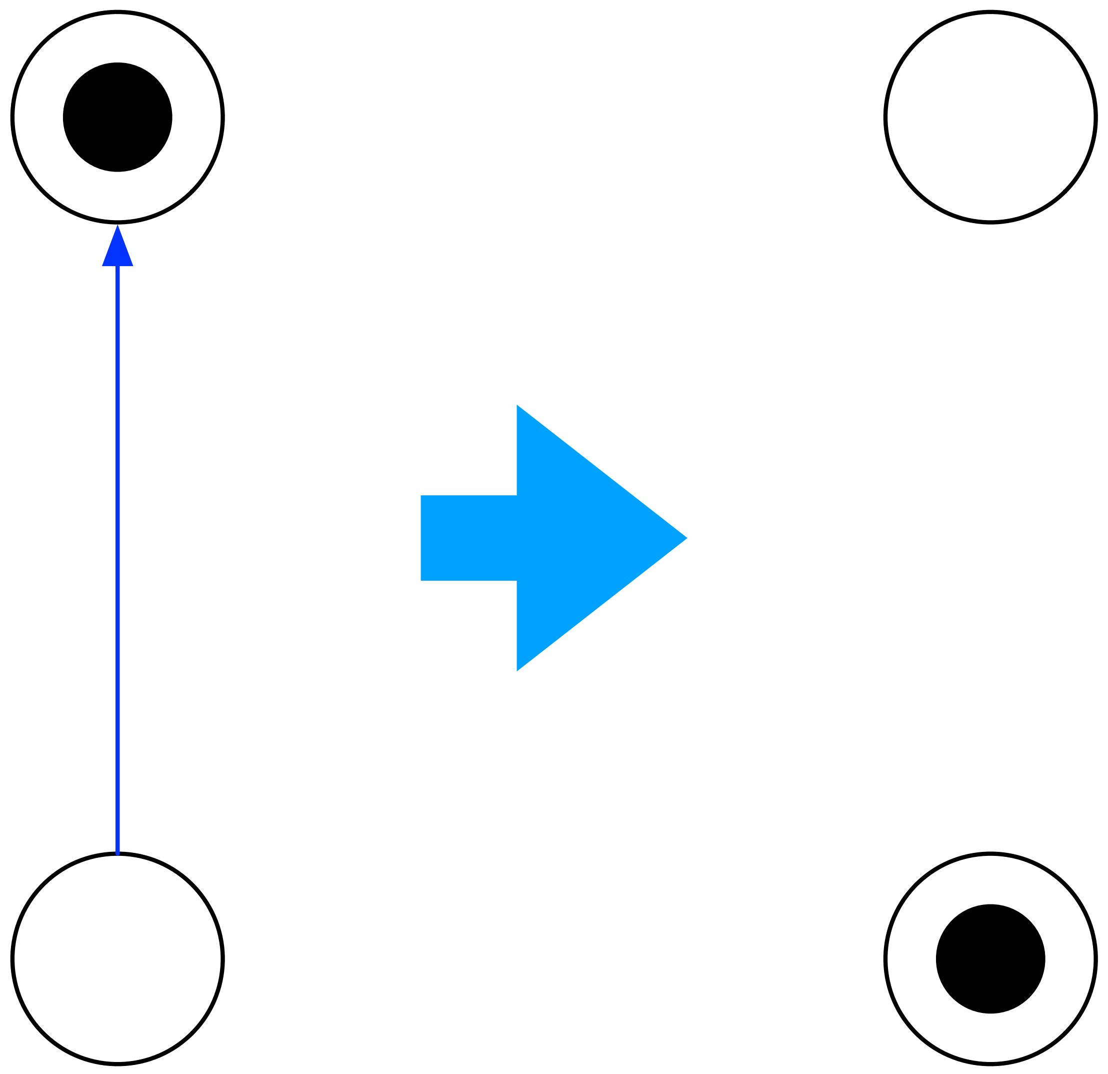


Token Dropping Game

- We are given an **directed acyclic graph**
- Each **node** holds either **0 or 1 token**
- Tokens can be **moved** from **u** to **v** if and only if:
  - the edge **{u, v} exists**
  - the edge **{u, v} is oriented from v to u**
  - **u** is holding a **token**
  - **v** is **not** holding a **token**
- Once a token **passes** through an edge, the edge **disappears**
- We want to reach a **stable** solution



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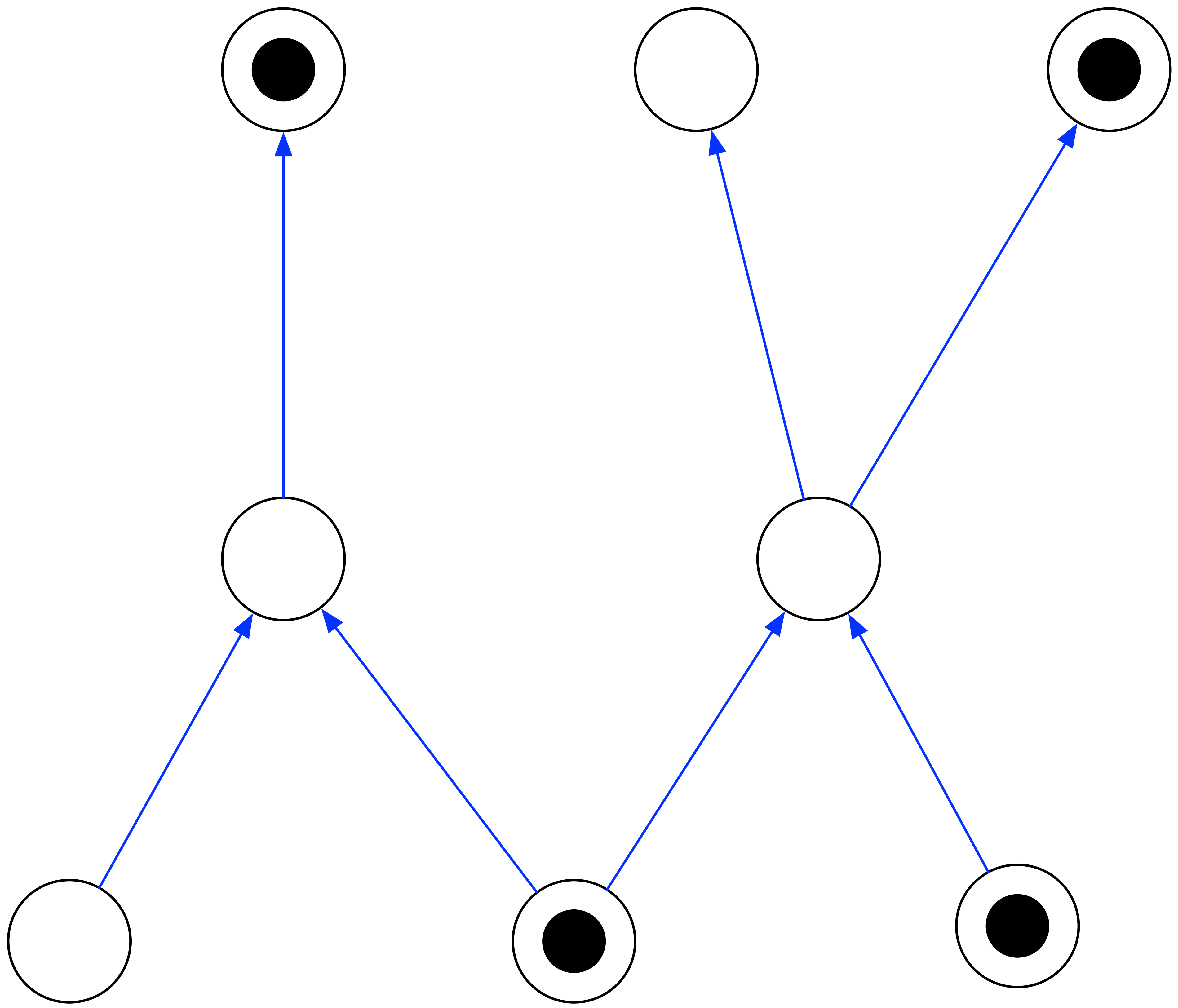

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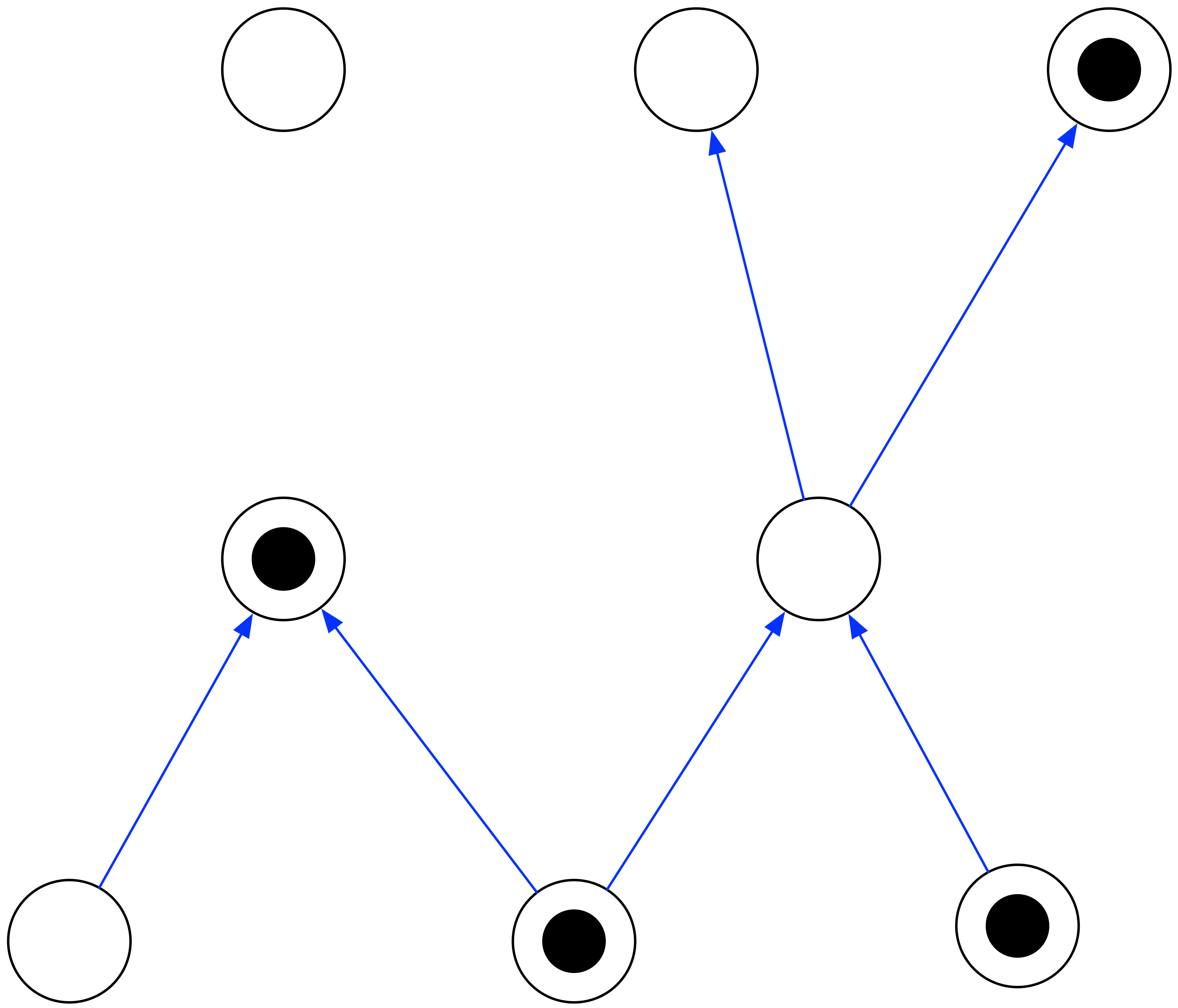
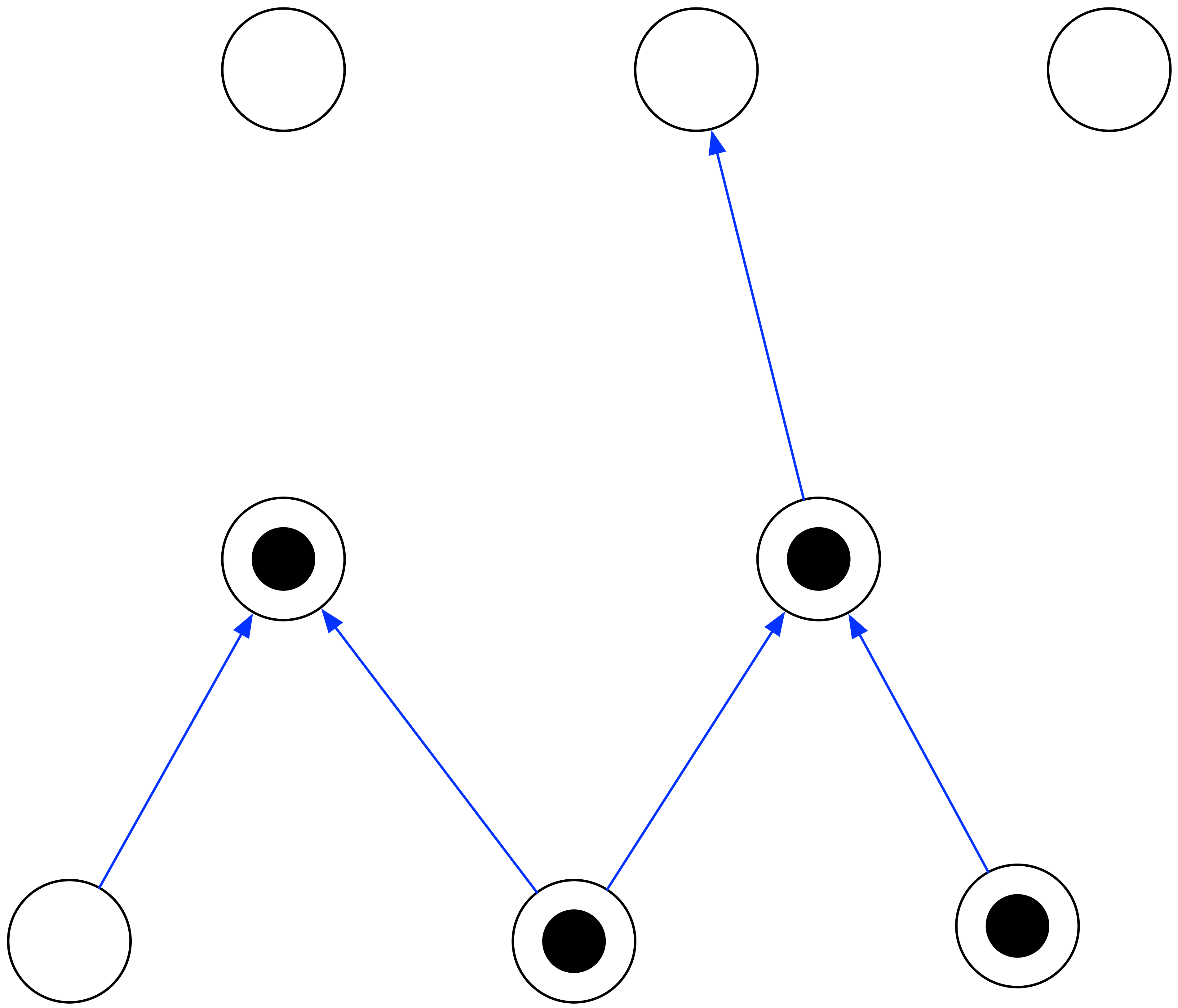
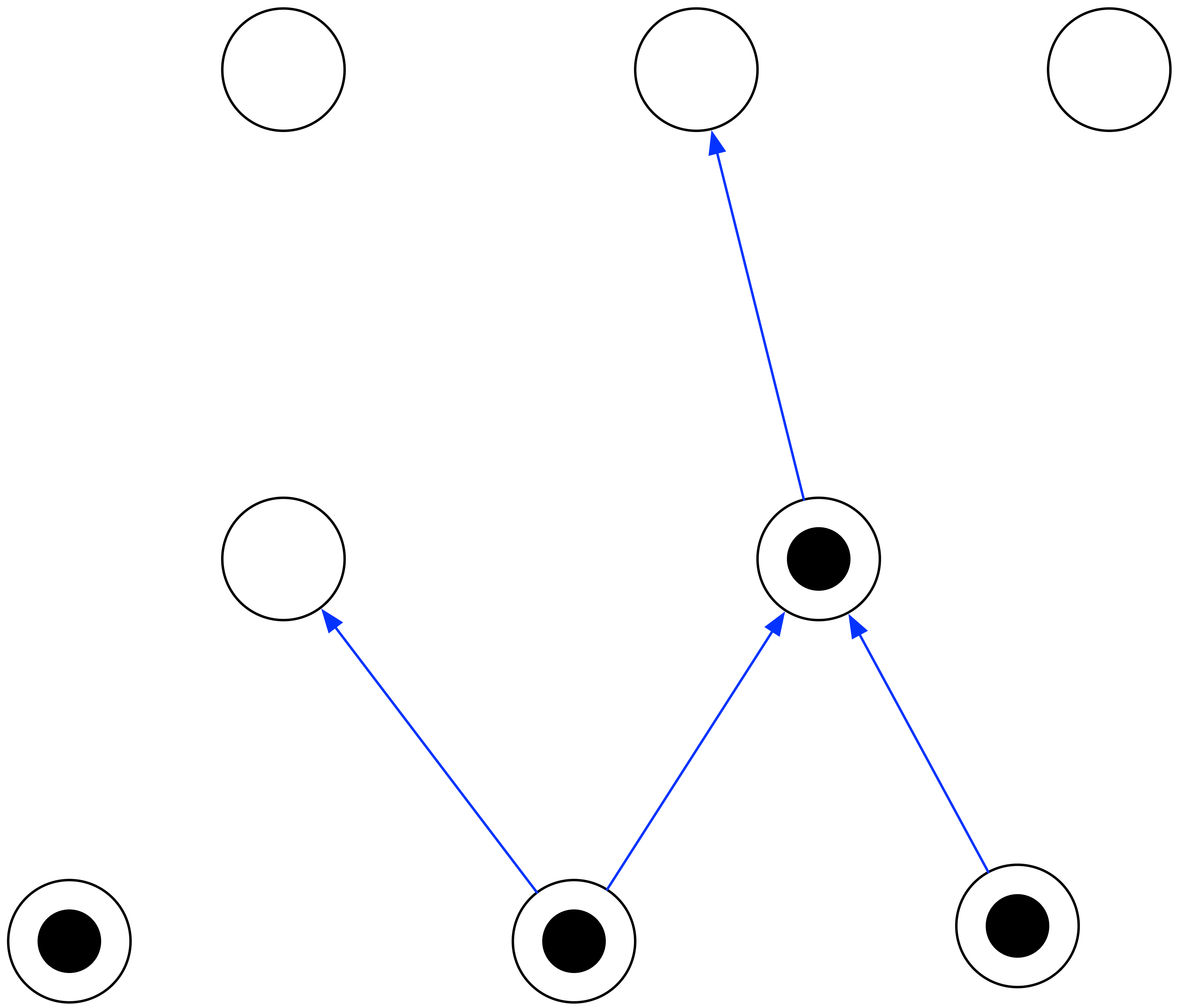


Token Dropping Game



Token Dropping Game

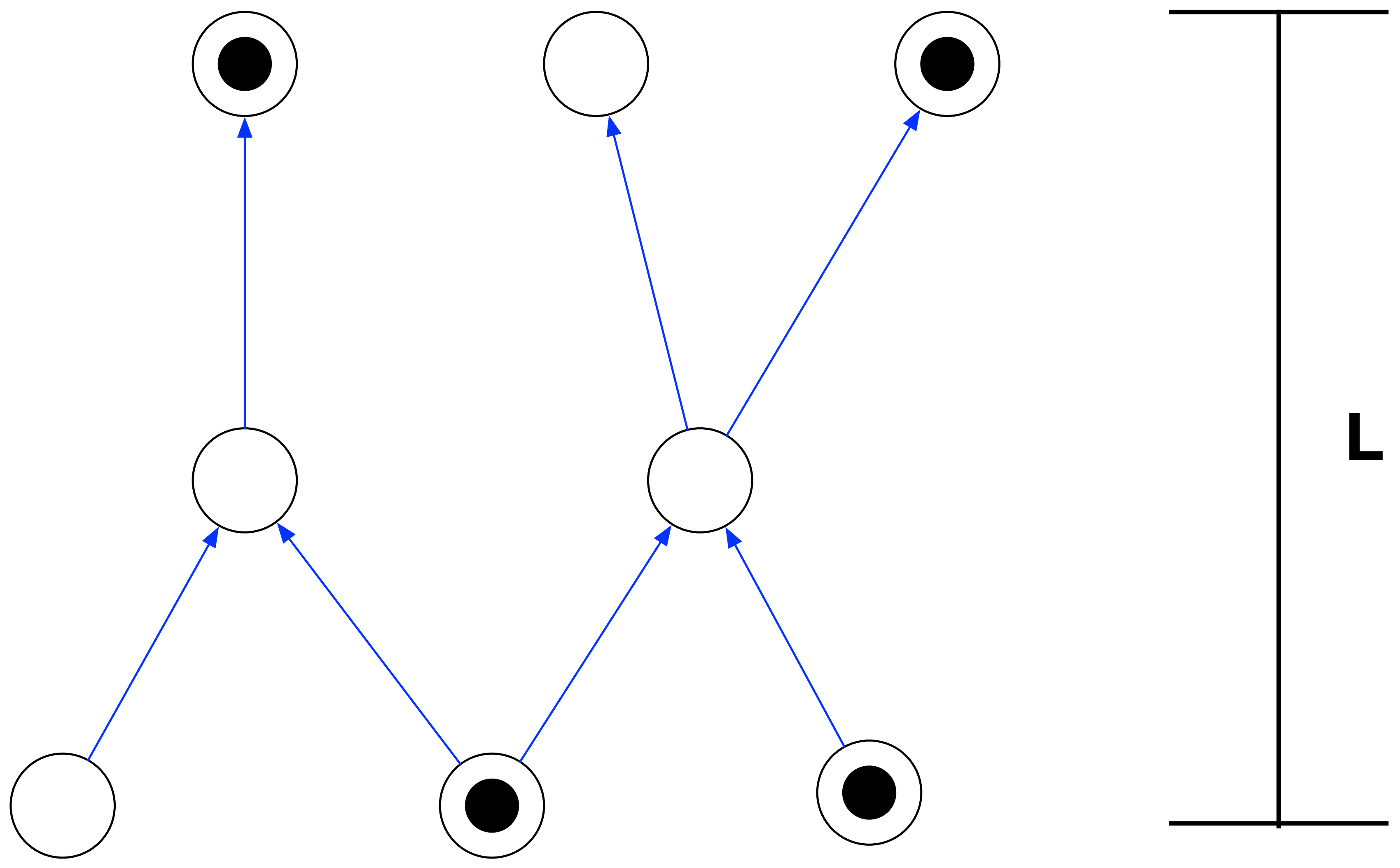


Token Dropping Game

Efficient Load-Balancing through Distributed Token Dropping

[Brandt, Keller, Rybicki, Suomela, Uitto 2021]

The Token Dropping Game can be solved in $O(L \cdot \Delta^2)$ rounds!



Semi Matching

Efficient Load-Balancing through Distributed Token Dropping

[Brandt, Keller, Rybicki, Suomela, Uitto 2021]

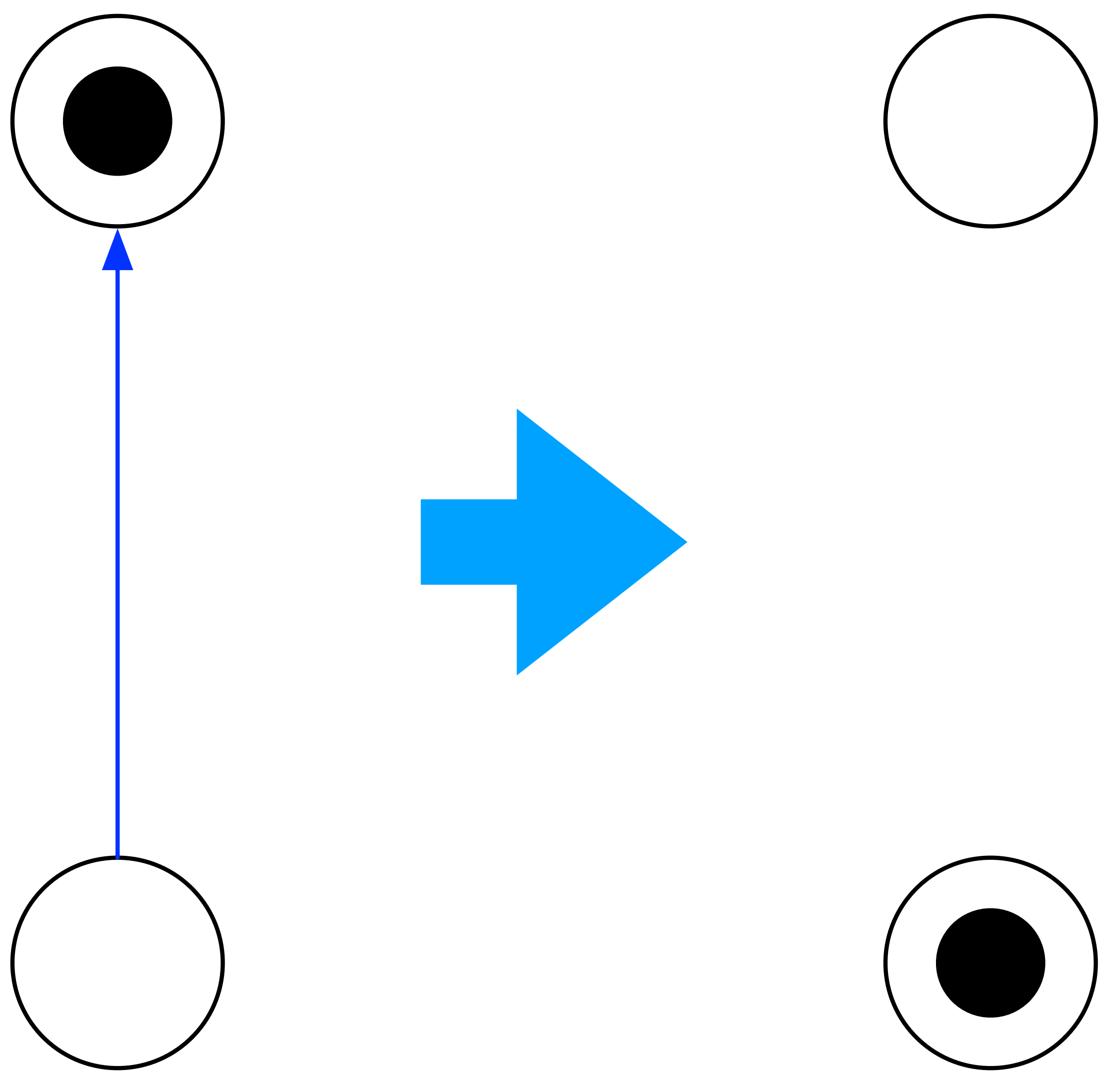
The Token Dropping Game can be solved in $O(L \cdot \Delta^2)$ rounds!

The Semi Matching problem can be solved by solving the Token Dropping Game for $O(\Delta)$ times (and $L = O(\Delta)$)

Semi Matching problem can be solved in $O(\Delta^4)$ rounds!

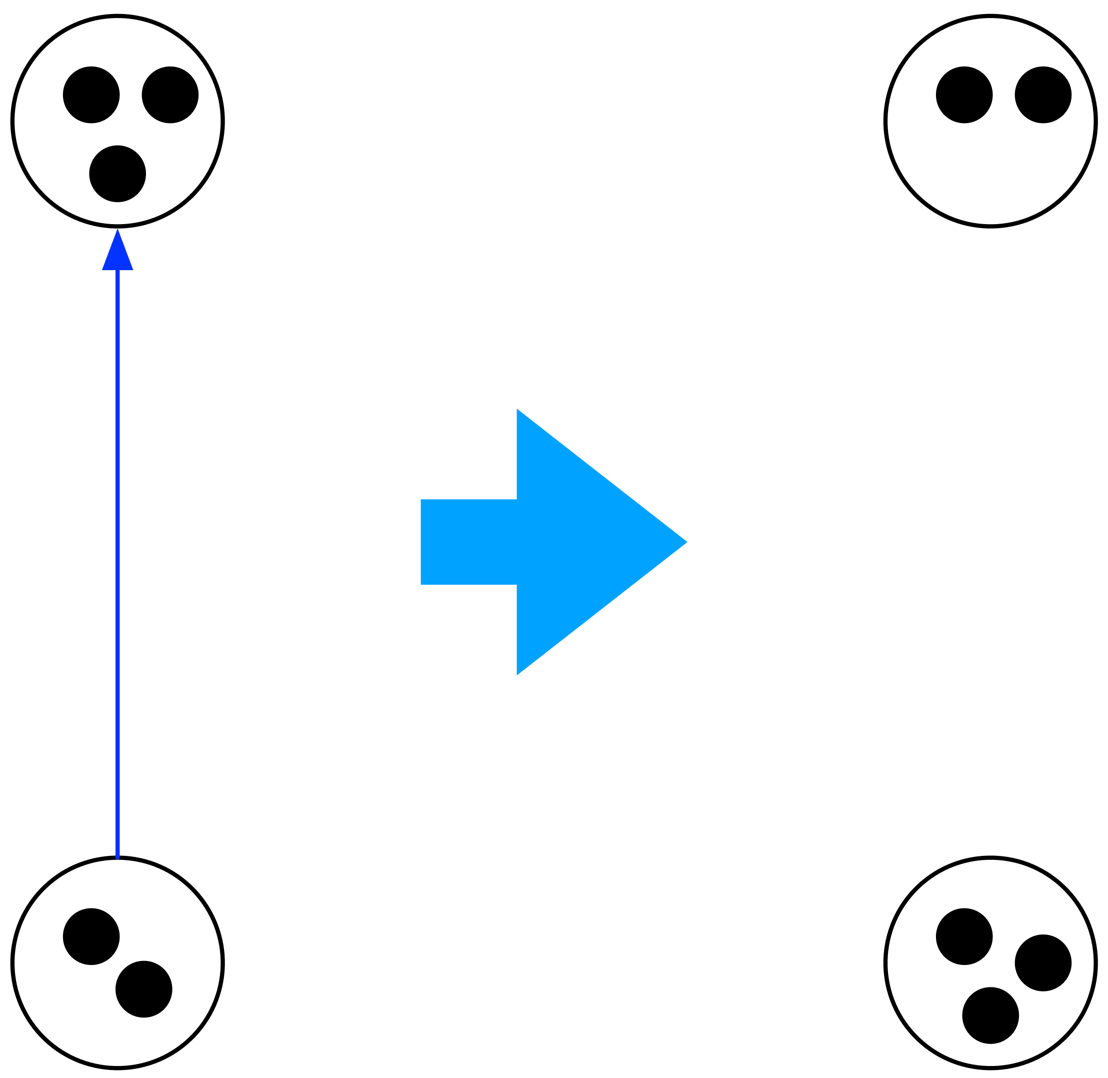
Token Dropping Game

- We are given an **directed acyclic graph**
- Each **node** holds either **0 or 1 token**
- A token can be **moved** from **u** to **v** if and only if:
  - the edge **{u, v} exists**
  - the edge **{u, v} is oriented from v to u**
  - **u** is holding a **token**
  - **v** is **not** holding a **token**
- Once a token **passes** through an edge, the edge **disappears**
- We want to reach a **stable** solution



Relaxed Token Dropping Game

- We are given an **directed acyclic graph**
- Each **node** holds **up to k tokens**
- A token **can** be **moved** from **u** to **v** if and only if:
  - the edge **{u, v} exists**
  - the edge **{u, v} is oriented from v to u**
  - **u** is holding **at least one token**
  - **v** is holding **at most k-1 tokens**, a token **must** move if **v** is holding **at most k/2** tokens
- Once a token **passes** through an edge, the edge **disappears**
- We want to reach a **stable** solution



Relaxed Token Dropping Game

Relaxed Token Dropping Game

- The Relaxed Token Dropping Game can be solved in $O(L \cdot \Delta^3/k^3)$ rounds!

Relaxed Token Dropping Game

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- The Relaxed Semi Matching problem can be solved in $O(\Delta^5/k^5)$ rounds!

Relaxed Token Dropping Game

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- For $k = \dfrac{\Delta}{\log \Delta}$, this gives an $O(\log^5 \Delta)$ round algorithm!

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- With some $\operatorname{poly}\log \Delta$ overhead, we can solve $O(\Delta)$-edge coloring!

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- For $k = \dfrac{\Delta}{\log \Delta}$, this gives an $O(\log^5 \Delta)$ round algorithm!

- With some $\operatorname{poly}\log \Delta$ overhead, we can solve $O(\Delta)$-edge coloring!

- For $(2\Delta - 1)$-edge coloring, things are harder. E.g., the game is not even on a DAG!

Open questions: upper bounds

- We can solve $(2\Delta - 1)$-edge coloring in $O(\log^{12} \Delta)$ + $O(\log^* n)$
  - Can we improve the exponent? We know a faster algorithm, but only for $O(\Delta)$-edge coloring
- Can we solve vertex coloring in subpoly(Δ)?

Open questions: lower bounds

- Can we prove a non-trivial lower bound for solving $(2\Delta - 1)$-edge coloring?
  - Can we show that it cannot be solved in $o(\log \Delta) + O(\log^* n)$?

Thank you!